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Mathematical Optimization for Challenging Network Planning Problems in Unbundled Liberalized Gas Markets

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Mathematical Optimization for Challenging Network Planning Problems in Unbundled Liberalized Gas Markets

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Abstract

The recently imposed new gas market liberalization rules in Germany lead to a change of business of gas network operators. While previously network operator and gas vendor where united, they were forced to split up into independent companies. The network has to be open to any other gas trader at the same conditions, and free network capacities have to be identified and publicly offered in a non-discriminatory way. We show that these new paradigms lead to new and challenging mathematical optimization problems. In order to solve them and to provide meaningful results for practice, all aspects of the underlying problems, such as combinatorics, stochasticity, uncertainty, and nonlinearity, have to be addressed. With such special-tailored solvers, free network capacities and topological network extensions can, for instance, be determined.

1 Introduction

In the year 2005 a new era began for companies operating in the German natural gas market. Back then a new Gas Network Access Regulation [7] was set in force. Being part of the European efforts in creating common energy markets for gas and electricity, the GasNZV describes the rules for a liberalized gas market in Germany. The central new aspect is the establishment of market areas based on an entry-exit system to ease network access and thus contribute to the realization of a competitive gas market. The former system forced gas shippers to book an entire transportation path through all the gas networks between the desired entry and exit points. This system has hindered that gas supply contracts have been signed across several areas in the past. With the new entry-exit system, capacity rights at entries and exits within a (large) market area can be booked by gas shippers independently of each other; they no longer need to care about the transportation path. It is the task of the gas network operator to enable all gas transportation requests within the limits of the booked entry and exit capacities. Moreover, the network operators are obliged to calculate and publish the spare capacity on the entries and exits of their networks so that gas shippers can book the offered capacities.

The new Gas Network Access Regulation brought a tremendous upheaval for the gas companies, since the European Union also demands the unbundling of the business unit operating the gas network from the remaining (usually vertically integrated) gas company [11]. These gas network operators are now companies on their own and operate separated from the holding company on all operational levels. Access to the gas network must be non-discriminatory, i.e., it has to be offered at the same terms to the holding company and to other gas shippers. The new entry-exit system and the monitoring of non-discriminatory network access by the regulation authorities lead to several new and challenging planning problems that have to be addressed for the first time. In fact, many of these emerging problems have very difficult mathematical optimization problems at their core. In this article we describe these problems from the network operator's perspective as well as from a mathematical point-of-view. We remark that there are further mathematical problems related to the natural gas industry, for example, in the production or the emergency management. For a broad survey of these problems we refer to Zheng et al. [40].

In the broadest sense mathematical optimization deals with the numerical computation of a proven optimum of a system that is described by a set of constraints and an objective function. The constraints are given as mathematical algebraic or differential equations over variables that describe, in our case, the physical properties of the gas, the technical properties of compressors, pipelines, or valves, or certain contractual or regulative situations. The goal is to assign values to variables such that the constraints are fulfilled, i.e., a feasible solution is achieved. Some variables can take values from a continuum, such as those for pressures and flows. Other variables are only allowed to take discrete values in order to represent the state of compressors and valves, where a zero means "closed" and a one means "open". If several alternative feasible solutions exist then they are rated by the objective function. The goal is now to find the best feasible solution with respect to this rating. When formulating mathematical models for gas networks one ends up with very large models: A network having a few hundred pipe segments and dozens of active network elements can easily lead to models having several thousand variables and constraints. These models have discrete and continuous variables, stochastic and deterministic data, and linear as well as nonlinear, nonconvex constraints. Identifying an optimal, or even any feasible solution is a computationally difficult process. Although the progress in terms of faster computer hardware and algorithms (or "solvers") for standard optimization problems over the last decades is tremendous (see Bixby [6]), all problems described in this article cannot be solved by just taking commercial standard solvers on modern computers. In fact, there does not even exist a solver for the problems we aim at.

Our scientific contribution is to identify the mathematical nature of the problems that are inherent in the laws and regulations, and that now govern the operational business of gas network operators. We characterize the problems and describe the difficulties when solving them using numerical standard solvers. Some of the problems can be reformulated, so that they are within reach of these solvers. Some solvers can be enhanced and specially tailored for our problem classes. Other problems however remain challenging after all, so that they provide a basis for future research.

Our work is supported by Open Grid Europe GmbH (OGE), which operates the former E.ON / Ruhrgas network. OGE provided real-world problem data which we used to develop models and algorithms.

The remainder of this article is organized as follows. In Section 2 we describe the current usage of the gas network at the entry and exit side, discussing the stochastic and uncertain behavior as well as the contractual situation. These insights can be used to estimate future usage of the network. In Section 3 we describe a mathematical model for the feasible states of a network with active and passive elements, such as compressors, valves, and pipelines. The efficient numerical solution of this model is the cornerstone for all subsequent tasks. The perhaps most elementary task is the validation of a nomination, that is, to decide if a given entry-exit situation leads to a physically and technically valid flow of gas in the network. We address this problem in Section 4. Next in line is the verification of booked capacities, which asks whether all entry-exit situations within the limits of the booked capacity rights can technically be realized. This question, that is closely related to the technical capacity of a network, is discussed in Section 5. If the booked capacities have been verified, then one can ask how much more free capacity the operator can offer to gas suppliers. We describe the mathematical implications of this question in Section 6. If the capacity does not meet the demand then it is necessary to enhance the network by building new facilities, such as compressors or pipelines. We utilize our model to this end in Section 7. An outlook to further challenges, such as the merging of formerly independent market regions, as well as conclusions, are given in Section 8.

2 Gas Networks Usage and Utilization

Obviously, gas consumers want to buy gas as cheap as possible. Gas vendors, on the other hand, want to reach as many consumers as possible. In an ideal market both requirements could be easily fulfilled; however, the situation on the gas market is more involved. The reason is that gas networks are, due to their high cost, a natural monopoly. Therefore, each customer receives gas from a single or just a few gas network operators, which in the past meant he had to buy gas via these network operators.

To change this situation and to ultimately establish a competitive market as mandated by the European Union [11], German regulatory authorities established an entry-exit system for gas network access: Ideally, capacity rights at entries and exits within a so-called *market area* can be booked and used independently of each other. The idea is that a gas consumer can satisfy his demand by buying gas from any gas vendor and in particular does not need to know where the gas comes from. The gas vendor, in turn, has to have sufficient capacity rights at the entry he intends to use, but does not need to know where the gas is withdrawn.

Using a gas network for transporting gas is a two-step process. First, a gas shipper needs to book capacity rights from the gas network operator that entitle him to feed in and/or feed out gas at specified points (bookable entries or exits) of the network subject to certain conditions. To use the capacity rights, a gas shipper has to *nominate* the amount of gas he intends to transport, usually a day before the actual transport takes place. Moreover, he has to ensure that the gas flow is *balanced*, i.e., the gas fed into the network is also fed out within a given time window. It is important that payment is for the capacity booked, not for the capacity used.

A gas network operator may offer different capacity products. The most favorable type of capacity from a market point of view is the so-called *freely allocable capacity (FAC)* at an entry or exit point. The amount of gas nominated on FAC at an entry point may be balanced with any exit point within the same market area, i.e., the network operator guarantees that the gas can be fed into the network at the entry and the same amount of gas can be withdrawn at the exit. However, the network operator does not know about which entry nomination balances which exit nomination. There is also *restrictively allocable capacity (RAC)*, for which the entry-exit combination of the gas flow is fixed. Moreover, there is *firm* and *interruptible* capacity. For firm capacity, the network operator is obliged to ensure that no matter how these capacities are actually used (i.e., how much is nominated), the resulting gas flows can technically be realized in the gas network under any condition. In contrast, the network operator may shorten the amount of gas transported, if it has been nominated on interruptible capacity contracts and this is necessary to operate the gas network under secure and safe conditions. Due to technical limitations of the gas network, the FAC that can be guaranteed by the network operator may be rather low at some entries or exits. In this case, the network operator may offer RAC or interruptible FAC to its customers.

A TSO is legally obliged to compute the maximum (firm) FAC at each point of his network and to offer this capacity to shippers. However, the flexibility of FAC is a major source of uncertainty for a gas network operator, since gas demand may be satisfied by any combination of entries. In general, it is not possible to anticipate which entries will be used, since this depends, among other things, on the gas price there. It is thus a delicate problem to check FAC availability and to maximize the FAC that can be booked.

Depending on the actual gas network, a certain share of the gas demand at exits may be modeled stochastically to some extent. Statistical data provided from measuring stations on exits within the German pipeline network operated by OGE gives a deeper insight into the stochastic properties and the behavior of the gas outflow. Typical exits in gas transmission networks are public utilities, industrial consumers and storages, as well as exits on borders and market crossings.

The dependence of gas loads on air temperature is observed usually in the case of public utilities and sometimes for industrial consumers, as illustrated in Figure 1. Statistical modeling techniques are convenient to model temperature dependent gas consumption, for details see Friedl et al. [14]. These statistical models can be

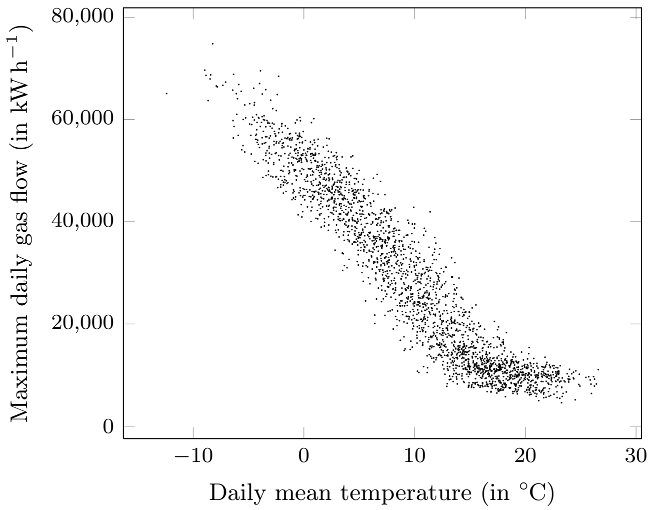

Figure 1: Example gas load of a public utility from 2004-01-01 to 2010-05-31.

applied to approximately 80 % of the exits in the network. For the remaining 20 % of exits statistical data is either unavailable or unreliable, as is the case for, e.g., storages.

3 Mathematical Modeling of Gas Networks

From a mathematical perspective a gas transmission network can be represented as a directed graph. The set of nodes consists of entries at which gas is supplied, exits at which gas is discharged and intermediate nodes. Arcs, comprising active and passive network elements, connect the nodes. Active elements such as compressor stations and different types of valves can be controlled by the network operator. Additional operational constraints restrict the interaction of these elements. In contrast, from a modelling perspective, operators do not have any influence on the behaviour of gas in passive network elements, namely pipelines, control valves without remote access, and resistors.

The physical behaviour of gas within a network can be modeled on different levels of detail. Common to all levels is that the flow of gas through network elements is coupled with gas pressure at the corresponding nodes of the network. Both, pressure and flow rates, are bounded by legal and technical constraints.

3.1 Nodes

Nodes are network elements without own capacity. Thus, a mass conservation (of Kirchhoff type) has to be satisfied at every node: The difference of flows on in- and outgoing arcs has to meet the nominated flow at the node.

3.2 Pipelines

Pipes are the most important components in gas transmission networks and typically outnumber all other network elements. They are often connected in series, forming pipelines that are used to transport gas over large distances. The gas flow through pipes is governed by the Euler equations for cylindrical pipes (see Feistauer [12] or Lurie [23]). Since we focus on gas network planning questions this system of partial differential equations is simplified by neglecting all transient effects yielding a system of ordinary differential equations (ODEs). This ODE system consists of the continuity and momentum equation. It is completed by an equation of state and a model for the real gas factor representing the deviation of ideal and real gas (see Modisette [25]).

The principal physical effect in pipes is the pressure loss due to friction at the rough inner pipe walls for turbulent flows. This effect mainly depends on the flow rate and the technical parameters of the pipe, e.g., its diameter, roughness, length and slope. If the gas temperature is considered as a dynamic variable, the pressure loss is additionally coupled by heat exchange effects with the surrounding soil. In this case, the ODE system is extended by the stationary energy equation modeling the addressed phenomena.

While the ODE system is a highly precise description of gas dynamics in pipes, it is not suitable for mathematical optimization. In the following sections approximations of ODE solutions (e.g., the Weymouth equation [39]) or discretizations of the ODE system will be used to model gas dynamics in pipes (see Section 4).

3.3 Resistors

Resistors are ficticious network elements for modeling pressure and temperature loss effects at obstacles, such as measuring and filter systems or complex piping facilities in compressor or control valve stations. There are no exact physical equations known for modeling gas dynamics for these obstacles. Thus, the pressure loss is approximatively modeled by a constant pressure decrease or, if more knowledge about the obstacle is given, by a pressure loss equation of Darcy–Weisbach type (see Lurie [23] or Finnemore and Franzini [13]).

3.4 Different Valve Types

Standard *valves* are active elements with two states: open or closed. If a valve is open, the pressure values at the end nodes coincide and arbitrary flow rates are allowed within some technical bounds. If a valve is closed, no gas can pass the valve and the pressures are decoupled.

A *control valve* is an extension of a valve. The open state is subdivided into an active and a bypass mode. If active, control valves can reduce the pressure within given technical bounds. A bypass mode leads the gas flow through a bypass inducing equal up- and downstream pressures. In consequence a control valve is either active, in bypass or closed.

A special subset of *control valves* are those *without remote access*. While the downstream pressure of general control valves can be controlled directly, these elements try to reduce the downstream pressure to a given threshold. When the upstream pressure drops below the threshold value, the control valve without remote access opens fully and thus is in bypass. If the downstream pressure rises above the threshold, the control valve without remote access closes automatically. As a result it is active, when the upstream pressure is above the threshold and the downstream pressure can be reduced to this value.

Both control valves with and without remote access have a fixed working direction. If the gas passes these units from the opposite direction, they have to be either closed or in bypass.

3.5 Compressor Stations

Compressor stations are used to increase gas pressure, which is necessary to transport gas over large distances. They are the most complex parts of gas transmission networks featuring both combinatorial switching of discrete states and highly nonlinear and nonconvex continuous machine models.

A compressor station consists of a set of compressor machines and drives which supply the compressors with power required for compression. For compressor machines we distinguish between turbo and piston compressors. Drives belong to one of several types of gas turbines, gas driven motors, electric motors and steam turbines. Both compressors and drives are specified by so-called *characteristic diagrams* determining the feasible operating range of the machines. For instance, Figure 2 shows an operating range of a turbo compressor defining the feasible combinations of up- and downstream pressure and flow.

Apart from these highly nonlinear machine models, different operation modes of the entire compressor station have to be handled. On the lowest level, each compressor may be active, bypassed, or closed. Several compressors might be used in parallel or in series, depending on the gas flow situation. Since compressor stations are often located at the intersection of several pipelines, gas may be transfered with or without compression from one pipeline to another. In the network model, a compressor station corresponds to a well-defined subnetwork consisting of compressors, short pipes without any pressure drop, and valves. The topology of the subnetwork is such that all operation modes can be realized by appropriate settings of the valves in this subnetwork. The set of feasible valve setting corresponding to the operation modes needs to be provided in addition to the network.

4 Validation of Nominations

A *nomination* is a vector giving the in- and outflow of gas at entry and exit nodes. While in reality supply and demand are not necessarily equal and can be compensated by a buffering network and gas market mechanisms, from our stationary point of view these time-dependent effects cannot be respected. Thus, the nomination vector is assumed to be balanced. In addition, a nomination implies certain technical and contractual bounds on the pressures on the entry and exit nodes. Furthermore, the supplied gas parameters, e.g., calorific values and other chemical

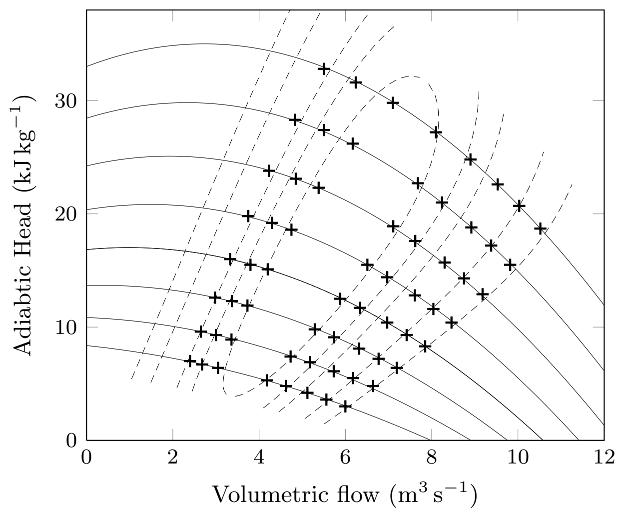

Figure 2: Operating range of a turbo compressor. The solid lines are the isolines for compressor speed and the dashed lines are the isolines for efficiency. Both sets of curves are chosen to fit the given measurements of the machine (+). The feasible operating range is defined by the maximum (top) and minimum (bottom) speed isolines and the leftmost and rightmost lines, i.e., the *surge*- and *chokeline*.

parameters like molar mass or gas density under normal conditions, are fixed at entry nodes.

The task of the *validation* of a nomination is to decide whether the gas network can be operated to fulfill the transportation request specified by the nomination, obeying technical and contractual limitations. This problem is frequently faced by gas network operators and is a major ingredient at all planning levels, operational but also strategical planning. To solve this problem in a constructive way, one has to compute controls for active network elements leading to physically and technically feasible values for gas pressures and flows. If a nomination can not be validated, a proof of infeasibility is desirable.

Translated to the language of mathematical optimization, the validation problem is a *nonlinear, non-convex* and *non-smooth, mixed-integer feasibility problem.* Directly solving this model for realistic, large-scale networks in full level of physical and technical detail is computationally intractable.

4.1 Current Solution Approaches of Gas Network Operators

Simulation is still the method of choice for many gas network planners. At this, all degrees of freedom are eliminated by fixing parameters and decisions, and the resulting system of nonlinear equations is then solved, e.g., by Newton-like methods. The main drawback of this approach is that the settings of the active network devices, in particular those who can not be controlled continously but are switched, have to be set manually by the network planners. Despite their experience, this is a highly time consuming work. There are a few tools available to automatically generate feasible solutions using heuristic methods to make educated guesses about the discrete decisions. Both simulation and the heuristic approaches have the drawback that they are not able to prove infeasibility.

4.2 From Power Contracts to Flow Problems

In practice, capacity contracts between gas network operators and gas shippers are stated in terms of power (kW). The network operator guarantees that the customer can supply or discharge gas with a certain limited amount of power. However, gas dynamics and the behaviour of technical elements are given in terms of (mass) flow ($\mathrm{kg\,s^{-1}}$). Since our model approaches have flow as their main variable, we transform a nomination stated in power into a nomination stated in flow. The transformation cannot be done exactly before solving the validation problem, i.e., before knowing the complete flow situation of the solution of the validation problem. Thus, we apply approximation techniques and assume a flow-based nomination to be given in the following.

4.3 A Two-Stage Approach for the Validation of a Nomination

To validate a nomination, we follow a two stage approach. First the complexity of gas physics and engineering are reduced while taking all discrete decisions into account. As a result, the first stage gives reasonable discrete controls of the active network elements together with a feasible state of continuous gas quantities like pressures and flows with respect to a simplified model.

Several solution approaches are applied to solve the first stage problem, see Section 4.4.

If one of the first stage approaches is successful, all determined discrete decisions are fixed and a purely continuous but highly accurate *nonlinear program* (NLP) is solved. This refines the coarse first stage solution so that feasibility of the nomination can be decided with respect to both discrete decisions and detailed gas dynamics and engineering models.

Due to the reduced complexity in each stage, both can be solved within reasonable runtimes.

4.4 Approaches to the First Stage

There exist various possibilities for obtaining a simplified first stage model, each with its specific benefits and drawbacks. We actually run four different approaches in parallel to obtain their combined benefits. These approaches, referred to as MILP, Spatial Branching (SB), RedNLP and MPEC, are briefly described in the following section. All approaches differ in terms of physical accuracy and in the techniques of handling the discrete model aspects. Since MILP and SB are based on branch-and-bound techniques, they are computationally expensive but are capable of proving infeasibility of a given nomination. RedNLP and MPEC are heuristics that are faster but cannot give provable statements on the given instance.

The MILP Approach The MILP approach (*mixed integer linear program*) is based on an algebraic isothermal stationary MINLP model (*mixed integer nonlinear program*) for gas flow. We build relaxations of the feasible set of the MINLP in terms of mixed-integer linear constraints only.

To this end we first construct a piecewise linear interpolation of each nonlinear expression such that the resulting approximation satisfies an a-priori given error bound. This is achieved by a new adaptive approximation algorithm based on convex underestimators (see Geissler et al. [15, 16]). In a second step we extend the so-called incremental method for piecewise linear functions to a MILP model for piecewise polyhedral outer approximations of the same tightness as the initially constructed approximation. Thus, the resulting MILP model is a proper relaxation of the underlying MINLP that incorporates all combinatorial constraints exactly and reflects any nonlinear constraint up to a predefined error bound.

In the MILP model characteristic diagrams of compressors are incorporated in terms of convex relaxations of their feasible ranges. In addition we apply an objective function that minimizes the infinity norm distance to the centroids of the characteristic diagrams of all active compressors in order to obtain a feasible control.

To solve the MILP relaxations any general purpose MILP solver can be chosen and in cases where the infeasibility of the MILP model is proven we also have an infeasibility proof for the underlying MINLP, due to the relaxation property.

The SB Approach The SB approach models the first stage problem as a non-convex mixed-integer constrained program which is solved by branch-and-bound.

The basis of the approach is the application of outer approximation techniques in combination with spatial branching. All nonlinear, non-convex equations are relaxed by a linear outer approximation which is refined by cutting planes. If cutting planes do not suffice to cut off points that are feasible in the relaxation but do not fulfill the equation, we branch on continuous variables to further improve the approximation of the nonlinear equation. This is referred to as *spatial branching*. The branch-and-bound algorithm is continued until a solution is found or infeasibility is proven. For more details we refer to Belotti et al. [2], Tawarmalani and Sahinidis [34, 35, 36], Smith and Pantelides [32], or Vigerske [37].

While this is the standard procedure for this kind of problem (see, for example, Berthold et al. [5]), we use the structure of the problem to improve the algorithm. If all integer variables take integer values, the remaining problem can be formulated as a convex program which can be solved to global optimality in reasonable running times. This is done whenever the relaxation is integral. If all integer variables are actually fixed, the solution of the convex problem suffices to prove feasibility or infeasiblity of the node and infeasible nodes can be pruned without any further spatial branching.

The RedNLP Approach The RedNLP approach (*reduced nonlinear program*) relies on transforming the model's nonlinearities into a more accessible form and to embed treatment of these transformed models into a heuristic procedure for finding promising switching decisions.

The system of (linear) flow conservation and (nonlinear) pressure drop equations is transformed into an equivalent nonlinear system where most flow and pressure variables get eliminated because they are explicit functions of a relatively small group of variables. This group consists of one flow variable per network cycle and one pressure variable at each node that is incident to an active element. Apart from the explicit formulas for flow and pressure variables the transformed system contains implicit equations whose number coincides with the number of fundamental cycles of the network and whose unknowns are just the variables from the mentioned group.

The idea behind this transformation dates back to at least the work of Hamam and Brameller [18] and has been picked up repeatedly later on by Mallinson and Fincham [24] and Rios-Mercado et al. [28]. Compared to this work, the approach taken here incorporates not only compressors, but also control valves and resistors. Moreover, it can handle substantially meshed gas distribution networks. It aims at checking feasibility for a set of switching states of active elements, either predefined or resulting from a transshipment heuristic. Any NLP solver can be used.

The MPEC Approach The MPEC heuristic (*mathematical program with equilibrium constraints*) handles the validation problem as a non-smooth MINLP and applies several techniques for transforming it into an NLP. Discrete controls of active network devices are modeled by *complementarity constraints* and non-smooth aspects are smoothed using both standard and model-specific smoothing techniques (see Schmidt [29] and Schmidt et al. [31]). This leads to an MPEC formulation. Due to a lack of model regularity, solving MPECs is a challenging task. Various model-specific regularization strategies are applied to address this difficulty. Finally, the resulting reformulation can be solved with standard NLP solvers.

As it is often the case, regularization and smoothing lead to numerically hard NLP formulations. To handle these difficulties, we split up the NLP solution process into two stages. The first stage deals with basic discrete controls of active network elements modeled by complementarity constraints. The subsequent stage fixes these decisions and attempts to find reasonable choices of compressor station configurations by a *convexification* of the set of all possible configurations.

4.5 The Second Stage: NLP Validation

As described above, all approaches for solving the first stage simplify the full problem of nomination validation. For this reason, we *validate* the solutions produced by the first stage approaches by a *highly detailed nonlinear feasibility problem* (see Schmidt et al. [30] for a detailed description of the model). The given solutions are used to fix the discrete controls of active devices and to initialize the nonlinear problem, which is finally solved by standard nonlinear optimization software.

4.6 Computational Results

We perform computational experiments on large-scale, real-world instances from our industrial partner OGE. The considered network is the northern part of their high-calorific gas network. It contains 452 pipes a total length of 1241 km, 9 resistors, 35 valves, 23 control valves and 6 compressor stations. A schematic plot is given in Figure 3. Up to our knowledge this is the first time that mathematical programming techniques have been successfully applied to gas network optimization problems of this size.

All experiments were conducted under Linux on a cluster where each node was equipped with two 3.2 GHz Intel Xeon X5672 quad-core CPUs and 48 GB RAM. Only one job at the time was submitted to each cluster node. A time limit of 2 hours was given. We use several commercial and open-source software packages to solve our models. Gurobi 5.0 [17] was used to solve the MILP problems constructed using the MILP approach. The SB approach is implemented in a prerelease version of the branch-and-cut solver SCIP 3.0 [1]. LP and NLP subproblems therein were solved using CPLEX 12.4 [20] and IPOPT 3.10 [38], respectively. Ipopt 3.10 was also used to solve the NLP problems in the RedNLP and the MPEC approach. Since the validation NLP can be tackled by several NLP solvers, we sequentially tried the solvers Ipopt 3.10, CONOPT 3.15C, CONOPT 4.00 [10], and KNITRO 8.0.0 [8] until one of them converge to a feasible point.

The four individual approaches were combined into a single nomination validation solver. On a parallel cluster they all solved the same instance of the problem simultaneously. If one of the four solvers is able to find a feasible solution, and this solution passes the NLP validation step, then this solution is returned to the user. In case one solver reports a feasible solution, and another solver proves infeasibility, the user will still obtain the solution, but together with warning that other solvers do not agree.

Based on statistical observations and the contractual situations at different points in time, we generated 4227 nominations (test set SN4). Our joint approach was able to solve 4157 (or more than 98%), that is, it either finds a feasible solution or it proves that the instances is infeasible. On 38 of the remaining 70 instances, the solvers came up with contradiction solutions, and on the final 32 instances no definitive result was produced, that means, all solvers neither found a solution nor proved infeasibility. The running times range from a few to 7200 seconds (i.e., reaching the time limit), taking 93 seconds on average and less than 21 seconds on half of the instances.

For more details, in particular on the individual behavior of the four solvers, we refer to our technical report [26].

5 Verifying Booked Capacities

As discussed in Section 2, regulatory authorities demand that a gas network operator offers as much capacity as possible as firm freely allocable capacity (FAC). However, the gas network operator may only offer firm capacity to the extent that all "likely and realistic" nominations within the booked capacities can technically be

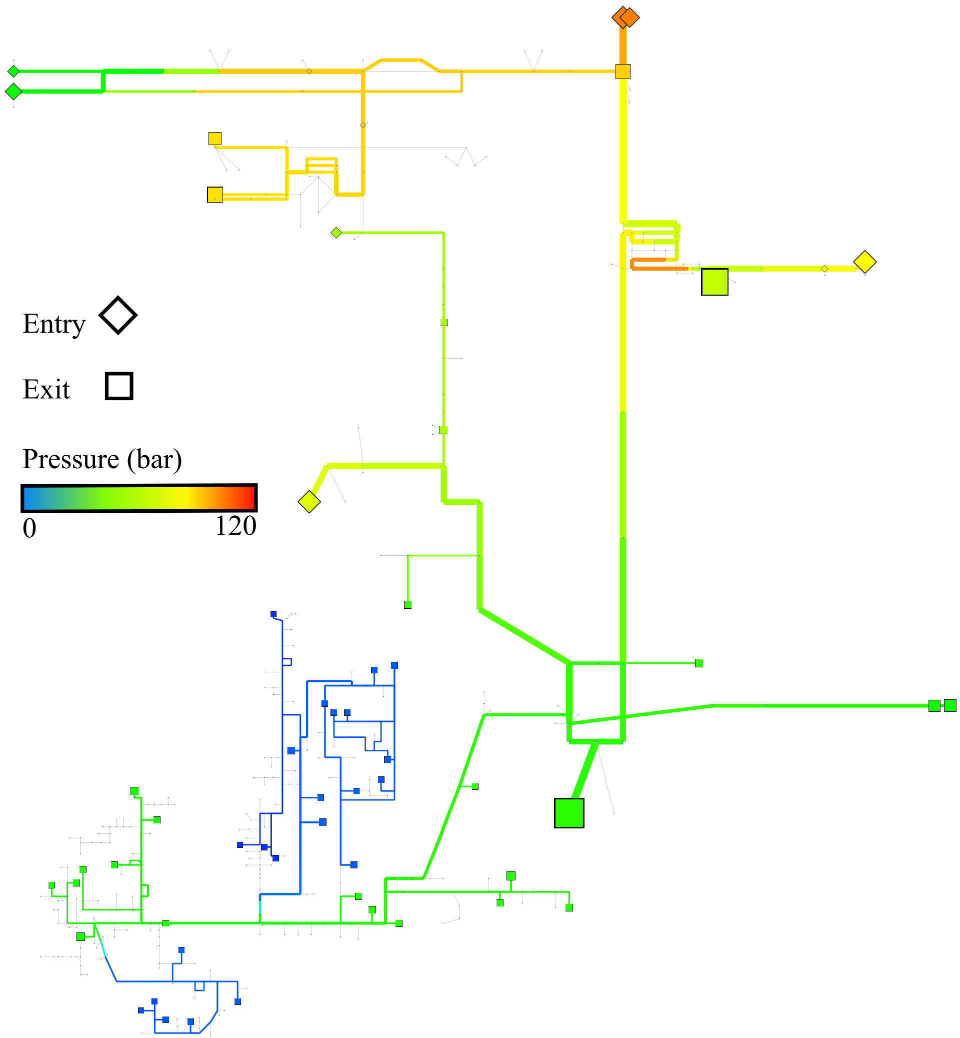

Figure 3: A sample solution for the realistic network used for computations. Width of lines represent the amount of flow. Colors represent the (average) pressure.

realized [7]. Gas network operators thus need a method to verify this realizability requirement.

We call a set of capacity products (e.g., firm FAC or RAC contracts) a *booking*. A nomination complies to this booking or is *booking-compliant*, if the nominated gas flows are in line with the rules of the capacity products. Most importantly, the flows at an entry or exit may not exceed the total booked capacity of this point.

The problem of verifying a booking may be viewed as a 2-stage 2-player game where the players are the gas network operator and a single customer representing all real customers who want to ship gas. In the first stage, the customer nominates a booking-compliant gas flow. The second stage allows the gas network operator to switch the components of the gas network in order to realize this gas flow. The network operator wins this game if and only if the gas flow can be realized, which means that the booking is valid. This type of problem has been discussed under different names in the literature. For instance, it is an example of an *adjustable robustness* feasibility problem (cf. Ben-Tal et al. [3]). It can also be readily expressed as a *quantified* feasibility problem, which is a special case of quantified optimization problems, see Subramani [33], Benedetti et al. [4], or Lorenz et al. [22]. "Quantified" in this context means that the problem formulation involves a sequence of logical quantifiers. Verification of a booking is a quantified version of the nomination validation problem: For *every* booking-compliant nomination *there is* a realizing setting of the active devices.

Requiring technical realizability for *every* booking-compliant nomination, no matter how unlikely, allows to offer only small capacities, since it is often easily possible to construct bad adversarial booking-compliant nominations. Moreover, it is also stricter than required legally. In practice, it is much more realistic to require that only a certain percentage (e.g., 95 %) of booking-compliant nominations needs to be technically realizable. To do this, one needs a stochastic model for the nominations. As discussed in Section 2, a stochastic model is appropriate only for a subset of the points, which we refer to as *statistical points*. The stochastic model defines a random vector providing the load at every statistical point. We call this random vector the *random load vector* and a realization of it a *statistical load scenario*. In the stochastic version of the problem, we require that with probability α, all booking-compliant nominations that extend a statistical load scenario are technically feasible. This formulation thus requires to determine the probability of an event described as a quantified feasibility problem.

The current industry standard to verify a booking is to check that a relatively small set of testing nominations is technically realizable. These testing nominations are chosen such that they are considered both challenging for the network but still realistic. To construct the testing nominations, one uses a combination of statistical information extracted from historical observations and knowledge about the structure and capabilities of the network. To check technical realizability, simulation and/or optimization tools are employed. However, in contrast to the method outlined in Section 4, none of the methods used is in principle able to reliably detect infeasibility.

Adjustable robust or quantified problem versions are usually much harder than their nominal problems and nomination validation is already a very hard problem.

Thus the development of an exact method for verifying a booking that is applicable to industrial-size networks is well beyond the reach of the current state-of-the-art. However, one can resort to an approximate approach that is similar to the one currently used in industry: In a first step, one generates a set of testing nominations, which are in the second step validated by a nomination validation method. The main reason for considering only few nominations is that it is labor-intensive to validate them manually. Once there is an effective automatic method for nomination validation (as the one presented in Section 4), there is no reason to stick to tenths of nominations. Depending on the running time for a single nomination validation, it may be feasible to test several thousands of nominations, thus increasing the confidence in the result of the booking verification significantly.

As in the existing method, we use stochastic models for the offtakes of the large part of the exits for which reliable statistical data is available and which are expected to behave in the future as they did in the past. We develop automatic methods that determine appropriate stochastic models and estimate their parameters from historical data. If statistically sound data is available, we model the joint behaviour of correlated exits by multivariate normal distributions. Otherwise, we use univariate normal and uniform distributions which may be *shifted*, i.e., there may be a positive probability that the offtake is exactly zero. In order to account for the dependence of the offtakes on the air temperature, these models are determined for several temperature classes separately. The temperature interval defining a temperature class is sufficiently small to treat the temperature as constant, thus removing the temperature dependency. The stochastic model is then used to generate statistical load scenarios, i.e., a joint set of loads on the statistical points, as described by Koch et al. [21]. These statistical load scenarios represent likely usage patterns for the network and thus capture the uncertainty of the load flows.

Since we need complete nominations, we also need to determine loads for entries and the remaining 20 % of the exits for which no stochastic model could be identified. These loads are generated by randomly sampling from all booking-compliant nominations that extend a given statistical load scenario. In this way, we obtain a large number (several thousands) of nominations. Finally, we use scenario reduction techniques (cf. Heitsch and Römisch [19]) to find a smaller number of nominations that are then validated. The use of scenario reduction allows to capture the underlying probability distribution well and to identify a small set of statistical load scenarios that approximately represent all original load scenarios.

The SN4 test set mentioned in Section 4.6 actually resulted from a proof-of-concept implementation for our approach to verify booked capacities. In a first step, we performed a statistical analysis of measured exit load data provided by OGE, comprising the years 2004–2010. Based on this data we created 14 temperature classes, usually with a width of 2 °C. The only exceptions are the border temperature classes $[-15\,°\text{C}, -4\,°\text{C}]$ and $[20\,°\text{C}, 40\,°\text{C}]$ which had to be chosen rather large due to lack of measurement data. For each temperature class we derived a stochastic model for a large part of the exit load flows and used this model to sample 1000 statistical load scenarios, which were reduced to 50 representative statistical load scenarios via scenario reduction [19]. Using a realistic,

temperature class	# nominations	validity probability α
$[20\,°C, 40\,°C]$	184	100 %
$[18\,°C, 20\,°C]$	186	97 %
$[16\,°C, 18\,°C]$	197	97 %
$[14\,°C, 16\,°C]$	193	98 %
$[12\,°C, 14\,°C]$	168	97 %
$[10\,°C, 12\,°C]$	286	97 %
$[8\,°C, 10\,°C]$	415	75 %
$[6\,°C, 8\,°C]$	430	70 %
$[4\,°C, 6\,°C]$	355	30 %
$[2\,°C, 4\,°C]$	295	9 %
$[0\,°C, 2\,°C]$	228	2 %
$[-2\,°C, 0\,°C]$	405	41 %
$[-4\,°C, -2\,°C]$	335	5 %
$[-15\,°C, -4\,°C]$	316	1 %

Table 1: Exemplary results of verifying a booking.

but rough model of booking-compliance for a subset of OGE's capacity contracts, we generated 100 complete nominations from each statistical load scenario, i.e., we randomly selected loads for the entries and remaining exits such that the entry power matches the exit power and all loads comply with our rough model of booking-compliance. These 100 nominations were reduced to a much smaller set of representative nominations that are sufficiently dissimilar according to a certain similarity measure.

These characteristics of the SN4 test set are shown in the first two columns in Table 1, giving the temperature class together with the number of SN4 nominations within it. As we can see, there are more dissimilar nominations in the lower temperature ranges than in the upper. The final column of Table 1 shows our estimate of the validity probability α of the considered booking, depending on the temperature class. This probability has been obtained by counting a statistical load scenario as verified, if and only if all of its nominations were feasible and then adding the probabilities of the verified statistical load scenarios. Due to the rough model for booking-compliance used so far and incompleteness of the contract data used, the given probabilities are far from the real ones and we show them here only to illustrate our approach and its results.

6 Available Freely Allocable Capacities

Gas transmission system operators are not only obliged to verify that booking-compliant transport situations can technically be realized, but they also have to offer the maximum amount of firm freely allocable capacity (FAC) that is technically possible. We call the problem of determining the maximum firm FAC the *capacity problem*.

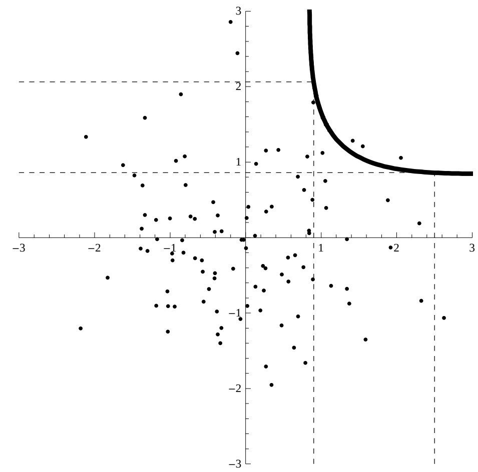

Figure 4: Illustration of the quantile set (thick curve) for a probability level $p = 0.8$ and a bivariate standard Gaussian distribution (represented here by 100 simulated points). The negative orthant attached to a quantile (dashed lines) covers realizations of the random vector with probability p (i.e., approximately 80 of the simulated points).

Formally, the capacity problem can be seen as an optimization variant of the former problem of verifying a booking as described in Section 5. Whereas for verifying a booking a probability needs to be determined, the capacity problem now features a chance constraint requiring that the probability that every booking-compliant nomination extending a statistical load scenario is technically feasible is at least α. Moreover, the capacities at different points of the network are physically interdependent: An increase of firm capacity at one point can decrease the firm capacity at another point. Necessarily, the determined capacities have to be a Pareto-optimum: capacity cannot be increased at any point unless it is reduced at another one.

Verifying a booking is the special case of the capacity problem where the vector of firm FAC is fixed. If the solution of the capacity problem is given in terms of the Pareto set, one can immediately verify each booking using this information. On the other hand, if the verification of a booking is not successful, i.e., the desired probability level is not reached, the capacity problem can be used to determine how much the firm FAC have to be reduced in order to achieve the probability level.

The fact that the capacity problem is an optimization variant of verifying a booking suggests that we extend the method presented in Section 5. Since, however, verifying a fixed booking is already very time consuming, there is little hope to realize it numerically in such an optimization setting. One way to circumvent this difficulty consists in aggregating the available stochastic information on the statistical points in so-called multivariate quantiles. A multivariate p-quantile is an artificial load scenario which (when viewed as a realization of the random load vector) dominates any statistical load scenario with probability p. Such quantiles are not unique (see Fig. 4) and they can be numerically determined as level sets to multivariate distribution functions (see Prekopa [27]). The latter are extracted from the statistical information as explained in Section 5. Based on this concept, a considerable numerical simplification of the capacity problem can be obtained under the following *monotonicity assumption*: if a certain statistical load scenario is feasible in the sense that all booking-compliant nominations extending it are technically feasible, then the same holds true for all statistical load scenarios which are component-wise smaller than the given one. As a consequence of this assumption and of the definition of a p-quantile one gets the following: if a p-quantile is feasible in the sense that all booking-compliant nominations extending this quantile are technically feasible, then the same holds true with probability at least p (e.g., $p = \alpha$) for any statistical load scenario. Therefore, instead of checking a large number of statistical load scenarios in order to verify the probability level α (as done when verifying a booking as described in Section 5), one may restrict the considerations to one or a small number of α-quantiles. Though some precision in the characterization of the probability level may get lost due to violation of the monotonicity assumption, this approach allows to capture the chance constraint of the capacity problem with an acceptable numerical effort.

In order to determine an approximation to the Pareto-set of firm FAC vectors for which the desired probability level can be achieved, we solve a relaxed variant of the capacity problem. We assume that for each point at which firm FAC should be offered, we know an upper bound on the firm FAC that is requested or can be used there. We subdivide the space spanned by these upper bounds, the *FAC space*, in small parts. For each of these parts, we try to establish feasibility for at least one nomination in each of these parts using a nomination validation tool as described in Section 4. However, instead of validating a fixed nomination, we allow intervals at each FAC point corresponding to the current part of the FAC space. In effect, a nomination validation tool then tries to find a nomination within this FAC interval that extends the given quantile and is technically feasible. If all nominations in a part are infeasible, bounds on the available capacity can be derived. We assume that if a part is "small enough", actually *every* nomination in this part is technically feasible.

7 Topology Planning

The topology planning starts with an infeasible nomination and has to determine network extensions such that the nomination can be realized in the extended network.

All network elements are available as extensions. Together with suitable locations, the dimensioning of the added elements has to be determined, e.g., the diameter of a new pipe or the power of a new compressor. Since in principle any two points can be connected by a pipe, there is a continuum of possibilities to extend the network.

Each extension is associated with construction costs. The construction cost of a new pipe can be estimated from the length of the pipe, its diameter and its exact course in the landscape. The course of the pipe is chosen with regard to the cost of different soil conditions that influence the construction process and costs for the purchase of land use rights. Some areas, such has highly populated areas or nature reserves, may not be traversed for pipeline construction. To this end, we apply Dijkstra's algorithm [9] to compute shortest paths on a grid graph with discretized geographical data. Extension pipes that run in parallel to existing pipes are known as *loops* and have reduced construction costs. A compressor incurs both construction and operating costs depending on its power, and control valves possess investment costs increasing with their flow capacity.

Our approach to topology planning subdivides the solution process into several steps, yielding a cost-effective extended network that is able to handle the nomination.

In a first step, we identify reasons for the infeasibility. This is done by adding slack variables to some constraints of the nomination validation problem. Non-zero slack translates to a violation of the respective constraints such that the absolute slack values are to be minimized. One possibility is to relax the flow conservation constraints, allowing gas to artificially enter or leave the network at any point.

Next, the results of the bottleneck analysis are used to obtain network extension candidates. An important task is to construct sets of extensions that already ensure feasibility of the nomination under consideration. The result of this second phase is a list of additional network elements.

In the next step, a cost-optimal subset of the extensions is selected. To this end, we extend the original network by the proposed extension elements. On this extended network, we solve a modified nomination validation problem where the usage of the extension elements is penalized with the elements construction and discounted operating costs.

For typical topology planning runs on the network shown in Figure 3, our implementation is already able to select a suitable subset of extensions from 50–70 candidates within a running time of 12 hours. This running may seem long, but is sufficient for strategic planning.

In practice such investment decisions would not be made on the basis of only one infeasible nomination. However, this approach can be extended to handle multiple nominations simultaneously ensuring robustness of the solution.

8 Outlook and Conclusions

The research on the topics described above is far from finished. First of all, the sizes of networks that should be optimized are increasing constantly. The regulation authority has increased the sizes of the *market areas* recently. Due to this and due to mergers in the gas transport operating companies aggregation of two or more networks is a future topic.

As more and more networks are aggregated the question arises whether the infrastructure is adequate to the required capacities. Furthermore, the security of supply is the topmost priority in gas transport. As networks grow bigger and the existing capacities are ever more utilized it becomes essential to have better planning tools based on accurate models. Topology planning could move into network transformation, i.e., at the same time adding and removing elements from the network.

Finally, so far no storage has been taken into account in the models we presented. This is a substantial deficiency. Devising models and optimization methods that can handle a semi-transient situation including storage facilities would be a major step forward.

We have no doubt gas networks will yield challenging questions for mathematicians for at least another decade.

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