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Generalized stochastic processes in algebras of generalized functions

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ABSTRACT

Stochastic processes with paths in a generalized function algebra are defined and it is shown that there exists an embedding of generalized functional stochastic processes into such ones. Gaussian stochastic processes with paths in an algebra of generalized functions are characterized by their first and second moments and an application to stochastic differential equations is given.

1. Introduction

We describe in this note stochastic processes, especially Gaussian ones, with paths in generalized function algebras through their representation with respect to a parameter $\varepsilon < 1$ and appropriate estimates with respect to $\varepsilon \to 0$. Our main motivation is to develop a basic calculus for singular stochastic processes, such as the white noise process, within an algebra of generalized stochastic processes. We will refer to stochastic processes with paths in generalized function algebras shortly as to GFA-stochastic processes. Also, we suggest a simple method of solving stochastic differential equations in the framework of Gaussian GFA-stochastic processes.

Generalized random processes were introduced by Gel'fand and Vilenkin in [3] and investigated by many authors. Depending on the type of *generalization* one can distinguish different types of generalized random processes. Generalized stochastic processes as elements of $L(V, L^2(\mathfrak{O}))$ i.e. as linear continuous mappings of a test space V into the space of random variables with finite second moments can be found in [9,10], while mappings into generalized random variable spaces with chaos expansions are subject of [6,7,11]. (In this paper we use the notation $\mathfrak{O}$ for the probability space and Ω for an open set of $\mathbb{R}^d$.) Another approach, originated from the older papers [5,23,24] (see also [12,13]), deals with generalized random processes defined as mappings $\xi : \mathfrak{O} \times V \to \mathbb{C}$ such that for every $\varphi \in V$, $\xi(\cdot, \varphi)$ is a complex random variable, and for every $\omega \in \mathfrak{O}$, $\xi(\omega, \cdot)$ is an element in V'. We have recently studied generalized processes of both types in [20] and [21].

Stochastic processes with paths in algebras of generalized functions are considered by Oberguggenberger, Russo and their coauthors in [1,15,17–19,22] and are used in solving some classes of nonlinear stochastic equations. The algebra of generalized functions considered in these papers is the Colombeau algebra of generalized functions denoted by $\mathcal{G}(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is an open set. This algebra contains the space of Schwartz distributions $\mathcal{D}'(\Omega)$ as a subspace. Stochastic processes are defined as mappings $\mathfrak{O} \to \mathcal{G}(\Omega)$ and called Colombeau generalized stochastic processes. Also, Oberguggenberger and Russo have shown that distribution stochastic processes i.e. weakly measurable mappings $X : \mathfrak{O} \to \mathcal{D}'(\Omega)$, where $X : \omega \mapsto \langle X(\omega), \phi\rangle$ is measurable for every $\phi \in \mathcal{D}(\Omega)$, can be embedded into Colombeau generalized stochastic processes. We refer

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also to [7] for different classes of generalized stochastic processes among which the Colombeau setting of such processes is mentioned.

In this paper we will consider stochastic processes with values in $L^p(\mathfrak{O})$, $p \geqslant 1$, and $\mathcal{L}(\mathfrak{O})$, the space of complex measurable functions endowed with almost sure convergence. For this purpose we introduce in Section 2 algebra $\mathcal{G}(\Omega, \mathcal{L}(\mathfrak{O}))$, vector spaces $\mathcal{G}(\Omega, L^p(\mathfrak{O}))$ and $\mathcal{G}(\Omega, \mathcal{L}(\mathfrak{O}))$, $\mathcal{G}(\Omega, [L^p(\mathfrak{O})])$ and call their elements GFA-stochastic processes. In Section 3 we analyze the embedding of spaces of generalized functional stochastic processes $L(\mathcal{D}(\Omega), L^p(\mathfrak{O}))$ and $L_0(\mathcal{D}(\Omega), \mathcal{L}(\mathfrak{O}))$ into the GFA-stochastic process space $\mathcal{G}(\Omega, L^p(\mathfrak{O}))$ and the GFA-stochastic process algebra $\mathcal{G}(\Omega, \mathcal{L}(\mathfrak{O}))$, respectively. In Section 4 we determine Gaussian GFA-stochastic processes, elements of $\mathcal{G}(\Omega, [L^p(\mathfrak{O})])$, by their first two moments which are elements of the algebra of generalized functions; in this approach, the classical proof is implemented with parametric estimates. As an application of this theorem, the white noise process is analyzed in the GFA-stochastic process setting. Moreover, solutions of a class of stochastic partial differential equations (SPDEs) in the framework of Gaussian GFA-stochastic process are analyzed and a suitable necessary condition for the existence of a solution is formulated.

2. GFA-stochastic processes

2.1. Algebra of generalized functions

We will deal with the algebra of generalized functions first constructed by Colombeau. Omitting the general construction [2,4,15], we recall only the definition of the algebra $\mathcal{G}(\Omega)$ on an open set Ω in $\mathbb{R}^n$. In order to simplify the understanding of this algebra one can have in mind distributions regularized through Friedrich's mollifiers and then extend this space of nets of regularized functions to a differential algebra of nets of functions. We will use the notations $D^\alpha = D_x^\alpha = \partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$ for the partial differential operator, $\alpha \in \mathbb{N}_0$.

Definition 2.1. Set $\mathcal{E}(\Omega) = (\mathcal{C}^\infty(\Omega))^I$, $I = (0, 1]$,

$$\mathcal{E}_M(\Omega) = \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{E}(\Omega)\colon\ (\forall K \Subset \Omega)\big(\forall \alpha \in \mathbb{N}_0^n\big)(\exists p \in \mathbb{N})\Big(\sup_{x\in K}\big|\partial^\alpha u_\varepsilon(x)\big| = \mathcal{O}\big(\varepsilon^{-p}\big)\Big)\Big\},$$

$$\mathcal{N}(\Omega) = \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{E}(\Omega)\colon\ (\forall K \Subset \Omega)\big(\forall \alpha \in \mathbb{N}_0^n\big)(\forall q \in \mathbb{N})\Big(\sup_{x\in K}\big|\partial^\alpha u_\varepsilon(x)\big| = \mathcal{O}\big(\varepsilon^{q}\big)\Big)\Big\},$$

$$\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega).$$

Elements of $\mathcal{E}_M(\Omega)$ and $\mathcal{N}(\Omega)$ are called moderate, resp. negligible nets of functions.

In the definition the Landau symbol $a_\varepsilon = \mathcal{O}(b_\varepsilon)$ appears, having the following meaning: $(\exists C > 0)(\exists \varepsilon_0 \in I)(\forall \varepsilon \in (0, \varepsilon_0))(a_\varepsilon \leqslant C b_\varepsilon)$.

Note that $\mathcal{E}_M(\Omega)$ is a differential algebra with pointwise operations. It is the largest differential subalgebra of $\mathcal{E}(\Omega)$ in which $\mathcal{N}(\Omega)$ is a differential ideal. Thus, $\mathcal{G}(\Omega)$ is an associative, commutative differential algebra. If $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega)$ is a representative of $u \in \mathcal{G}(\Omega)$, we write $u = [(u_\varepsilon)_\varepsilon]$.

We use a net of mollifiers $\varphi_\varepsilon \in \mathcal{S}(\mathbb{R}^n)$, $\varepsilon \in I$, of the form

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^n,\ \varepsilon \in I, \tag{1}$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ has the properties $\int \varphi(x)\,dx = 1$, $\int x^k \varphi(x)\,dx = 0$, $k \in \mathbb{N}$, and φ is positive definite i.e. $\hat{\varphi} \geqslant 0$, where $\hat{\varphi}$ denotes the Fourier transformation of φ. (For example one can take $\hat{\varphi} \in \mathcal{D}(\mathbb{R}^n)$, $\hat{\varphi} \geqslant 0$ and $\hat{\varphi} \equiv 1$ in a neighborhood of zero.) The Fourier transformation of $\phi \in \mathcal{S}(\mathbb{R}^n)$ is defined as $\hat{\phi}(x) = \int_{\mathbb{R}^n} e^{-i\langle x,t\rangle}\phi(t)\,dt$, and the Fourier transformation in $\mathcal{S}'(\mathbb{R}^n)$ is given by $\langle \hat{f}, \phi\rangle = \langle f, \hat{\phi}\rangle$, $f \in \mathcal{S}'(\mathbb{R}^n)$, $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Let be a compactly supported Schwartz distribution, $T \in \mathcal{E}'(\Omega)$. Then by

$$T \rightsquigarrow Cd(T) = \big[\big((T * \varphi_\varepsilon)|_\Omega\big)_\varepsilon\big] = \big((T * \varphi_\varepsilon)|_\Omega\big)_\varepsilon + \mathcal{N}(\Omega)$$

is defined a linear embedding of $\mathcal{E}'(\Omega)$ into $\mathcal{G}(\Omega)$. Since the presheaf $\Omega \mapsto \mathcal{G}(\Omega)$ is a soft sheaf, it follows that the above embedding can be extended to an embedding of $\mathcal{D}'(\Omega)$ and $C^\infty(\Omega)$ into $\mathcal{G}(\Omega)$ for any open set $\Omega \subset \mathbb{R}^n$.

Rather than considering equality in $\mathcal{G}(\Omega)$, it is more usual to consider the notion of association. We say that $u = [(u_\varepsilon)_\varepsilon]$ and $v = [(v_\varepsilon)_\varepsilon]$ are associated in $\mathcal{G}(\Omega)$, denoted by $u \approx v$, if

$$\lim_{\varepsilon\to 0}\int_\Omega \big(u_\varepsilon(x) - v_\varepsilon(x)\big)\phi(x)\,dx = 0, \quad \text{for all } \phi \in \mathcal{D}(\Omega).$$

Similarly, $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$ is associated with an element $f \in \mathcal{D}'(\Omega)$ (in this case f is called the distributional shadow of u) if $\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon(x)\phi(x)\,dx = \langle f, \phi\rangle$, for all $\phi \in \mathcal{D}(\Omega)$.

2.2. Definitions of GFA-stochastic processes

As we already stated, $(\mathfrak{O}, \mathfrak{U}, P)$ denotes a probability space, and $\Omega \subset \mathbb{R}^d$ is an open set.

Definition 2.2. Let $\mathcal{E}(\Omega, \mathcal{L}(\mathfrak{O}))$ be the set of nets $(u_\varepsilon(\omega, x))_\varepsilon$, $\omega \in \mathfrak{O}$, $x \in \Omega$, $\varepsilon \in I$, such that, for almost every (a.e.) $\omega \in \mathfrak{O}$, it holds $(u_\varepsilon(\omega, \cdot))_\varepsilon \in (\mathcal{C}^\infty(\Omega))^I$, and for every $x \in \Omega$, $(u_\varepsilon(\cdot, x))_\varepsilon$ is a net of measurable functions on $\mathfrak{O}$. Set

$$\mathcal{E}_M\big(\Omega, \mathcal{L}(\mathfrak{O})\big) = \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{E}\big(\Omega, \mathcal{L}(\mathfrak{O})\big):\ (\text{for a.e. } \omega \in \mathfrak{O})(\forall K \Subset \Omega)\big(\forall \alpha \in \mathbb{N}_0^n\big)(\exists a \in \mathbb{N})\Big(\sup_{x\in K}\big|\partial^\alpha u_\varepsilon(\omega, x)\big| = \mathcal{O}\big(\varepsilon^{-a}\big)\Big)\Big\},$$

$$\mathcal{N}\big(\Omega, \mathcal{L}(\mathfrak{O})\big) = \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{E}\big(\Omega, \mathcal{L}(\mathfrak{O})\big):\ (\text{for a.e. } \omega \in \mathfrak{O})(\forall K \Subset \Omega)\big(\forall \alpha \in \mathbb{N}_0^n\big)(\forall b \in \mathbb{N})\Big(\sup_{x\in K}\big|\partial^\alpha u_\varepsilon(\omega, x)\big| = \mathcal{O}\big(\varepsilon^{b}\big)\Big)\Big\}.$$

Elements of $\mathcal{E}_M(\Omega, \mathcal{L}(\mathfrak{O}))$ and $\mathcal{N}(\Omega, \mathcal{L}(\mathfrak{O}))$ are called moderate and negligible nets of functions with values in $\mathcal{L}(\mathfrak{O})$, respectively.

The space of $\mathcal{L}(\mathfrak{O})$ GFA-stochastic processes is defined as

$$\mathcal{G}\big(\Omega, \mathcal{L}(\mathfrak{O})\big) = \mathcal{E}_M\big(\Omega, \mathcal{L}(\mathfrak{O})\big)/\mathcal{N}\big(\Omega, \mathcal{L}(\mathfrak{O})\big).$$

As $\mathcal{E}_M(\Omega, \mathcal{L}(\mathfrak{O}))$ is an algebra with respect to multiplication, and $\mathcal{N}(\Omega, \mathcal{L}(\mathfrak{O}))$ an ideal in $\mathcal{E}_M(\Omega, \mathcal{L}(\mathfrak{O}))$, we have that $\mathcal{G}(\Omega, \mathcal{L}(\mathfrak{O}))$ is an algebra.

The next step is to adopt the theory of generalized stochastic processes to the case of function with values in $L^p(\mathfrak{O})$.

Definition 2.3. Let $p \geqslant 1$. Denote: $\mathcal{E}(\Omega, L^p(\mathfrak{O})) = (\mathcal{C}^\infty(\Omega, L^p(\mathfrak{O})))^I$. Then:

$$\mathcal{E}_M\big(\Omega, L^p(\mathfrak{O})\big) = \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{E}\big(\Omega, L^p(\mathfrak{O})\big):\ (\forall K \Subset \Omega)\big(\forall \alpha \in \mathbb{N}_0^n\big)(\exists a \in \mathbb{N})\Big(\sup_{x\in K}\big\|\partial^\alpha u_\varepsilon(\cdot, x)\big\|_{L^p} = \mathcal{O}\big(\varepsilon^{-a}\big)\Big)\Big\},$$

$$\mathcal{N}\big(\Omega, L^p(\mathfrak{O})\big) = \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{E}\big(\Omega, L^p(\mathfrak{O})\big):\ (\forall K \Subset \Omega)\big(\forall \alpha \in \mathbb{N}_0^n\big)(\forall b \in \mathbb{N})\Big(\sup_{x\in K}\big\|\partial^\alpha u_\varepsilon(\cdot, x)\big\|_{L^p} = \mathcal{O}\big(\varepsilon^{b}\big)\Big)\Big\}.$$

Elements of $\mathcal{E}_M(\Omega, L^p(\mathfrak{O}))$ and $\mathcal{N}(\Omega, L^p(\mathfrak{O}))$ are called moderate and negligible nets of functions with values in $L^p(\mathfrak{O})$, respectively.

The space of $L^p(\mathfrak{O})$ GFA-stochastic processes is defined as

$$\mathcal{G}\big(\Omega, L^p(\mathfrak{O})\big) = \mathcal{E}_M\big(\Omega, L^p(\mathfrak{O})\big)/\mathcal{N}\big(\Omega, L^p(\mathfrak{O})\big).$$

Let $(\mathfrak{O}, \mathcal{L}_\varepsilon(\mathfrak{O}), P_\varepsilon)$, $\varepsilon \in I$, be a net of probability spaces and let $L^p_\varepsilon(\mathfrak{O})$, $\varepsilon \in I$, be a net of corresponding L^p-spaces. We will consider in Section 4 the following extension of Definition 2.3.

Definition 2.4. Let $p \geqslant 1$. Denote: $\mathcal{E}(\Omega, [L^p_\varepsilon(\mathfrak{O})]) = (\mathcal{C}^\infty(\Omega, L^p_\varepsilon(\mathfrak{O})))^I$. Then:

$$\mathcal{E}_M\big(\Omega, \big[L^p_\varepsilon(\mathfrak{O})\big]\big) = \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{E}\big(\Omega, \big[L^p_\varepsilon(\mathfrak{O})\big]\big):\ (\forall K \Subset \Omega)\big(\forall \alpha \in \mathbb{N}_0^n\big)(\exists a \in \mathbb{N})\Big(\sup_{x\in K}\big\|\partial^\alpha u_\varepsilon(\cdot, x)\big\|_{L^p_\varepsilon} = \mathcal{O}\big(\varepsilon^{-a}\big)\Big)\Big\},$$

$$\mathcal{N}\big(\Omega, \big[L^p_\varepsilon(\mathfrak{O})\big]\big) = \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{E}\big(\Omega, \big[L^p_\varepsilon(\mathfrak{O})\big]\big):\ (\forall K \Subset \Omega)\big(\forall \alpha \in \mathbb{N}_0^n\big)(\forall b \in \mathbb{N})\Big(\sup_{x\in K}\big\|\partial^\alpha u_\varepsilon(\cdot, x)\big\|_{L^p_\varepsilon} = \mathcal{O}\big(\varepsilon^{b}\big)\Big)\Big\}.$$

We use the notation $\mathcal{E}_M(\Omega, [L^p_\varepsilon(\mathfrak{O})])$, $\mathcal{N}(\Omega, [L^p_\varepsilon(\mathfrak{O})])$ as well as

$$\mathcal{G}\big(\Omega, \big[L^p_\varepsilon(\mathfrak{O})\big]\big) = \mathcal{E}_M\big(\Omega, \big[L^p_\varepsilon(\mathfrak{O})\big]\big)/\mathcal{N}\big(\Omega, \big[L^p_\varepsilon(\mathfrak{O})\big]\big)$$

and call them spaces of moderate and negligible nets of functions and nets of functions with values in $[L^p(\mathfrak{O})]$, respectively, as well as $[L^p(\mathfrak{O})]$ GFA-stochastic processes.

Clearly, if $L^p_\varepsilon = L^p$, $\varepsilon \in I$, then $\mathcal{E}_M(\Omega, [L^p_\varepsilon(\mathfrak{O})]) = \mathcal{E}_M(\Omega, L^p(\mathfrak{O}))$.

Remark 1. Note that by "push forward" all the ε-random variables $\mathfrak{O} \ni \omega \to u_\varepsilon(\omega, x) \in \mathbb{R}^n$ $(x \in \Omega)$ are pushed to $\mathbb{R}^n$ with the Borel σ-algebra and ε-weighted Lebesgues measures, $\varepsilon \in I$. In this sense, the definition of Gaussian GFA-stochastic processes in Section 4 is given without referring on a net $(L^p_\varepsilon(\mathfrak{O}))_\varepsilon$.

We emphasize that the operation of multiplication is not closed in $\mathcal{E}_M(\Omega, L^p(\mathfrak{O}))$ and $\mathcal{E}_M(\Omega, [L^p(\mathfrak{O})])$.

3. Embeddings of generalized functional stochastic processes into GFA-stochastic processes

3.1. Generalized functional stochastic processes

First, we recall in an abstract manner two classes of generalized stochastic processes, depending on the type of convergence. Let us denote by V a locally convex space and let V' be its dual space. Then, by $L(V,\mathcal{L}(\mathfrak{O}))$ and $L(V,L^p(\mathfrak{O}))$ are denoted the spaces of linear and continuous mappings from V into the respected spaces. Elements of these spaces are called generalized functional stochastic processes. In general case, if $\xi(\cdot,\varphi)\in L(V,\mathcal{L}(\mathfrak{O}))$, $\varphi\in V$, then one cannot claim continuity of the mapping $\xi(\omega,\cdot)$ for fixed $\omega\in\mathfrak{O}$.

We will consider in this paper $L(V,L^p(\mathfrak{O}))$ and $L_0(V,\mathcal{L}(\mathfrak{O}))\subset L(V,\mathcal{L}(\mathfrak{O}))$, where $L_0(V,\mathcal{L}(\mathfrak{O}))$ consists of those elements $f(\omega,\phi)$, $(\omega,\phi)\in\mathfrak{O}\times V$ which have the property: There exists $\mathfrak{O}_f$ a subset of $\mathfrak{O}$ of zero measure such that

$$f(\omega,\cdot)\in V' \quad \text{for every } \omega\in\mathfrak{O}\setminus\mathfrak{O}_f.$$

For example, if V is a nuclear space, then $L_0(V,\mathcal{L}(\mathfrak{O}))=L(V,\mathcal{L}(\mathfrak{O}))$ i.e. on nuclear spaces each generalized functional stochastic processes has a version with continuous paths (this was shown in [25]). Examples of spaces where $L_0(V,\mathcal{L}(\mathfrak{O}))\neq L(V,\mathcal{L}(\mathfrak{O}))$ can be found in [21].

Theorem 3.1. *Let V be a separable space and $\xi_1,\xi_2\in L_0(V,\mathcal{L}(\mathfrak{O}))$. The following conditions are equivalent.*

(a) *For every $\varphi\in V$, $P\{\omega\colon \xi_1(\omega,\varphi)=\xi_2(\omega,\varphi)\}=1$.*
(b) *There exists $W_0\subset\mathfrak{O}$, such that $P(W_0)=1$, and, for every $\omega\in W_0$, $\xi_1(\omega,\cdot)=\xi_2(\omega,\cdot)$ in V'.*

Proof. Let (a) hold and V_0 be a countable dense set in V. Set, for $\varphi\in V_0$, $W_\varphi=\{\omega\colon\ \xi_1(\omega,\varphi)=\xi_2(\omega,\varphi)\}$. Clearly, $P(W_\varphi)=1$. Denote

$$W_0=\bigcup_{\varphi\in V_0}W_\varphi=\bigcup\bigl\{\omega\colon\ \xi_1(\omega,\varphi)=\xi_2(\omega,\varphi),\ \ \varphi\in V_0\bigr\}.$$

Since $P(W_0)=1$, we have to prove (b) for any $\omega\in W_0$ and $\varphi\in V$. There exists a sequence $(\varphi_n)_n\in V_0^{\mathbb{N}}$, such that $\varphi_n\to\varphi$ in V, as $n\to\infty$. The equality $\xi_1(\omega,\varphi_n)=\xi_2(\omega,\varphi_n)$ for $\varphi_n\in V_0$, $\omega\in W_0$, implies that

$$\xi_1(\omega,\varphi)=\lim_{n\to\infty}\xi_1(\omega,\varphi_n)=\lim_{n\to\infty}\xi_2(\omega,\varphi_n)=\xi_2(\omega,\varphi).$$

The converse assertion is obvious. □

If $\xi_1,\xi_2\in L(V,L^p(\mathfrak{O}))$, then they are equal if for any $\varphi\in V$,

$$\int_{\mathfrak{O}}\bigl|\xi_1(\omega,\varphi)-\xi_2(\omega,\varphi)\bigr|^p\,dP(\omega)=0.$$

Let $\xi\in L(V,L^p(\mathfrak{O}))$. Since the inclusion mapping $L^p(\mathfrak{O})\to L^1(\mathfrak{O})$ is continuous, it follows that the linear mapping $V\ni\varphi\rightsquigarrow\int_{\mathfrak{O}}\xi(\omega,\varphi)\,dP(\omega)$ is continuous. Thus the expectation of ξ, $E(\xi(\omega,\varphi))=m(\varphi)$, $\varphi\in V$, exists and belongs to V'. Without loss of generality, we may assume (when needed) that $E(\xi(\cdot,\varphi))=0$, for an arbitrary $\varphi\in V$.

If $E(\xi(\omega,\varphi)\overline{\xi(\omega,\psi)})$ exists for all φ and ψ, and it is continuous with respect to each argument φ and ψ (for example, if $p=2$), the correlation operator of $\xi\in L(V,L^p(\mathfrak{O}))$ is defined by

$$B_\xi(\varphi,\psi)=E\bigl(\xi(\cdot,\varphi)\overline{\xi(\cdot,\psi)}\bigr),\quad \varphi,\psi\in V,$$

and the covariance operator is defined by

$$C_\xi(\varphi,\psi)=B_\xi(\varphi,\psi)-m(\varphi)m(\psi).$$

3.2. Embeddings

In the sequel, we put $V=\mathcal{D}(\Omega)$ endowed with the usual topology in $\mathcal{D}(\Omega)$. Let $\xi\in L_0(\mathcal{D}(\Omega),\mathcal{L}(\mathfrak{O}))$, respectively $\xi\in L(\mathcal{D}(\Omega),L^p(\mathfrak{O}))$. Denote by $(\kappa_\varepsilon)_\varepsilon$ a net of smooth functions supported by $\Omega_{-\varepsilon}=\{x\in\Omega\colon\ d(x,\mathbb{R}^n\setminus\Omega)>\varepsilon\}$, $\varepsilon\leqslant\varepsilon_0$, such that $\kappa_\varepsilon\equiv 1$ on $\Omega_{-2\varepsilon}$, $\varepsilon<\varepsilon_0$. Let $(\varphi_\varepsilon)_\varepsilon$ denote the net of mollifiers defined by (1). Then

$$u_\varepsilon(\omega,x)=(\kappa_\varepsilon\xi)\bigl(\omega,\varphi_\varepsilon(\cdot-x)\bigr),\quad \omega\in\mathfrak{O},\ \ x\in\Omega,\ \ \varepsilon\in I, \tag{2}$$

defines an element of $\mathcal{E}_M(\Omega,\mathcal{L}(\mathfrak{O}))$, respectively $\mathcal{E}_M(\Omega,L^p(\mathfrak{O}))$, denoted by $u=[(u_\varepsilon)_\varepsilon]$. If ξ is compactly supported, then in (2) we may take

$$u_\varepsilon(\omega,x)=\xi\big(\omega,\varphi_\varepsilon(\cdot-x)\big),\quad \omega\in\mathfrak{O},\ x\in\Omega,\ \varepsilon\in I,$$

since $(\kappa_\varepsilon\xi)(\omega,\phi)=\xi(\omega,\phi)$, $\omega\in\mathfrak{O}$, $\phi\in\mathcal{D}(\Omega)$, $\varepsilon\in I$.

We want to show that different generalized functional stochastic processes on $\mathcal{D}(\Omega)$ define different GFA-stochastic processes. Also, we shall give an example of a GFA-stochastic process which is not obtained by the embedding of a functional stochastic process.

First we quote some preassumptions we will use in the sequel. Every $\xi\in L_0(\mathcal{D}(\Omega),\mathcal{L}(\mathfrak{O}))$, respectively $\xi\in L(\mathcal{D}(\Omega), L^p(\mathfrak{O}))$ can be written in the form $\xi=\sum_{i=1}^\infty \xi\chi_i$, where $(\chi_i)_{i\in\mathbb{N}}$ is a partition of unity for an open cover of Ω consisting of bounded sets so that $K_i=\operatorname{supp}\chi_i\Subset\Omega$, $i\in\mathbb{N}$, and so we have $\xi\chi_i\in L_0(\mathcal{E}(\Omega),\mathcal{L}(\mathfrak{O}))$, $i\in\mathbb{N}$. Thus, without loss of generality in studying the equality of stochastic processes, we will assume that ξ has a compact support (else we use a partition of unity and the sheaf properties of $\mathcal{D}(\Omega)$). We will also assume that compact sets under consideration are of the form $\overline{W}$, where $W\subset\Omega$ is an open bounded set with a boundary that is regular enough, so that functions from $C^m(\overline{W})$, $m\in\mathbb{N}$, can be extended to $C^m(\Omega_1)$ for some Ω_1, $W\Subset\Omega_1\Subset\Omega$. (When W is an open set, the notation $W\Subset\Omega_1$ means that $\overline{W}$ is a compact subset in Ω_1.)

Now we quote some results of distribution theory. Let $\xi\in L_0(\mathcal{E}(\Omega),\mathcal{L}(\mathfrak{O}))$ and $K_0=\operatorname{supp}\xi$. Then, for almost all $\omega\in\mathfrak{O}$, $\xi(\omega,\cdot)$ can be continuously extended to functions from $C^m(K)$ for some $m\in\mathbb{N}$ and some compact set K, $K_0\Subset K\Subset\Omega$ (here we used the preassumptions quoted above). Now, for every $\phi\in C^m(K)$ there exists an open set Ω_1, $K\Subset\Omega_1\Subset\Omega$, so that ϕ can be extended to a function (again denoted by ϕ) belonging to $C^m(\Omega_1)$. Let Ω_2 be an open set such that $K\Subset\Omega_2\Subset\Omega_1$. The net of smooth functions $\Omega_2\ni x\mapsto\int_{\Omega_2}\varphi_\varepsilon(x-t)\phi(t)\,dt$, $\varepsilon<\varepsilon_0$, converges to ϕ in $C^m(K)$.

Lemma 2. *Let $(\varphi_\varepsilon)_\varepsilon$ be the net of mollifiers defined in* (1) *and $K\Subset\Omega_2$ as quoted above. The set of functions $A=\{\varphi_\varepsilon(\cdot-t)|_K;\ t\in\Omega_2,\varepsilon\in(0,1)\}$ is dense in $C^m(K)$.*

Proof. We will show that if $g(\omega,\cdot)\in(C^m(K))'$ for almost all $\omega\in\mathfrak{O}$, then for almost all $\omega\in\mathfrak{O}$:

$$g(\omega,\phi)=0,\quad \phi\in A\quad\Longrightarrow\quad g(\omega,\psi)=0,\quad \psi\in C^m(K).$$

This will imply the assertion. Let $\psi\in C^m(K)$ so that it can be extended in Ω_1. We have for almost all $\omega\in\mathfrak{O}$,

$$\langle g,\psi\rangle=\lim_{\varepsilon\to0}\left\langle g(\omega,\cdot),\int_{\Omega_2}\varphi_\varepsilon(\cdot-t)\psi(t)\,dt\right\rangle=\lim_{\varepsilon\to0}\big\langle\big\langle g(\omega,\cdot),\varphi_\varepsilon(\cdot-t)|_K\big\rangle,\psi(t)\big\rangle=0.$$

This completes the proof. □

Now, we prove the main results of this section, that (2) defines an embedding $L(\mathcal{D}(\Omega),L^p(\mathfrak{O}))\hookrightarrow\mathcal{G}(\Omega,L^p(\mathfrak{O}))$, as well as an embedding $L_0(\mathcal{D}(\Omega),\mathcal{L}(\mathfrak{O}))\hookrightarrow\mathcal{G}(\Omega,\mathcal{L}(\mathfrak{O}))$. In other words, both generalized functional stochastic processes with finite moments and generalized functional stochastic processes with continuous versions can be embedded into appropriate GFA-stochastic process spaces.

Theorem 3.2. *If $\xi_1,\xi_2\in L_0(\mathcal{D}(\Omega),\mathcal{L}(\mathfrak{O}))$ and $\xi_1\neq\xi_2$, then the corresponding elements u_1 and u_2 of $\mathcal{G}(\Omega,\mathcal{L}(\mathfrak{O}))$ are also different.*

Proof. Recall, we may assume that $\xi=\xi_1-\xi_2$ is compactly supported, else we use the partition of unity. Since, $\xi\neq0$, it follows that there exists a function $\varphi_0\in\mathcal{D}(\Omega)$ and a set $W_0\subset\mathfrak{O}$, $P(W_0)>0$, such that, for every $\omega\in W_0$, $\xi(\omega,\varphi_0)\neq0$. Let $K_0=\operatorname{supp}\xi$ and $K_0\Subset K\Subset\Omega$ as before. By Lemma 2, there exist a sequence $(t_n)_{n\in\mathbb{N}}$ in Ω_2 and a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,1)$, such that $\varphi_{\varepsilon_n}(\cdot-t_n)\to\varphi_0(\cdot)$ in $C^m(K)$ as $n\to\infty$. For every $\omega\in W_0$

$$\xi\big(\omega,\varphi_{\varepsilon_n}(\cdot-t_n)|_K\big)\to\xi(\omega,\varphi_0)\neq0,\quad n\to\infty.$$

Thus, for every $\omega\in W_0$,

$$\sup_{t\in\overline{\Omega}_2,n\in\mathbb{N}}\big|\xi\big(\omega,\varphi_{\varepsilon_n}(\cdot-t)\big)\big|\neq o(1),$$

which means that $\xi(\omega,\varphi_\varepsilon(\cdot-t))$ is not a negligible function, i.e. ξ_1 and ξ_2 determine different elements of $\mathcal{G}(\Omega,\mathcal{L}(\mathfrak{O}))$. □

Now we show that different elements of $L(\mathcal{D}(\Omega),L^p(\mathfrak{O}))$ determine different elements of $\mathcal{G}(\Omega,L^p(\mathfrak{O}))$.

Theorem 3.3. *Let $\xi_1,\xi_2\in L(\mathcal{D}(\Omega),L^p(\mathfrak{O}))$ and $\xi_1\neq\xi_2$. The corresponding elements u_1 and u_2 of $\mathcal{G}(\Omega,L^p(\mathfrak{O}))$ are also different.*

Proof. Assume again that $\xi=\xi_1-\xi_2$ has a compact support K_0. Since $\xi\neq0$, it follows that there exists $\varphi_0\in\mathcal{D}(\Omega)$ such that, for each $\omega\in\mathfrak{O}$, $\int_{\mathfrak{O}}|\xi(\omega,\varphi_0)|^2\,dP(\omega)\neq0$. By the same arguments and notations as in Theorem 3.2, we have that $\xi\neq0$ implies that $\xi(\cdot,\varphi_{\varepsilon_n}(\cdot-t_n)|_K)$ does not tend to zero in the sense of L^2-convergence as $n\to\infty$.

The set Ω_2 contains t_n, $n \in \mathbb{N}$. Now, from

$$\sup_{t\in\overline{\Omega}_2} \int_{\mathfrak{O}} \left|\xi\big(\omega, \varphi_\varepsilon(\cdot - t)\big)\right|^2 dP(\omega) \neq o(1) \quad \text{as } \varepsilon \to 0,$$

it follows that $u_{1\varepsilon} - u_{2\varepsilon} \notin \mathcal{N}(\Omega, L^p(\mathfrak{O}))$. This proves the theorem. □

Example 3.1. Consider $u_\varepsilon(\omega, x) = \frac{1}{\varepsilon^2}\varphi^2(\frac{x}{\varepsilon})X(\omega)$, $x \in \Omega$, $\varepsilon \in I$, where $X(\omega)$ denotes a random variable, resp., a random variable in $L^p(\mathfrak{O})$, and φ is the mollifier. This element is not obtained by the embedding of an element of $L(\mathcal{D}(\Omega), \mathcal{L}(\mathfrak{O}))$, resp., of $L(\mathcal{D}(\Omega), L^p(\mathfrak{O}))$. The associated element in $L(\mathcal{D}(\Omega), \mathcal{L}(\mathfrak{O}))$ would be $\delta^2(x)X(\omega)$, but we know that δ^2 does not exist in $\mathcal{D}'(\Omega)$.

4. $[L^2(\mathfrak{O})]$ GFA-stochastic processes

4.1. Expectation and correlation

Let $u \in \mathcal{G}(\Omega, [L^2_\varepsilon(\mathfrak{O})])$. The expectation of u is an element m of $\mathcal{G}(\Omega)$ whose representative is

$$m_{u_\varepsilon}(x) = E\big(u_\varepsilon(\omega, x)\big) = \int_{\mathfrak{O}} u_\varepsilon(\omega, x)\, dP_\varepsilon(\omega), \quad x \in \Omega,\ \varepsilon \in I.$$

The correlation function of u is an element B of $\mathcal{G}(\Omega \times \Omega)$ with a representative

$$B_{u_\varepsilon}(x, y) = E\big(u_\varepsilon(\omega, x)u_\varepsilon(\omega, y)\big) = \int_{\mathfrak{O}} u_\varepsilon(\omega, x)u_\varepsilon(\omega, y)\, dP_\varepsilon(\omega), \quad x, y \in \Omega,\ \varepsilon \in I.$$

Let $\xi \in L(\mathcal{D}(\Omega), L^p(\mathfrak{O}))$ and $u_\varepsilon(\omega, x)$ be defined by (2). Then the representatives of the expectation m_{u_ε} and the correlation function B_{u_ε}, as well as the process u_ε itself, depend on the choice of the mollifier function. However, they define elements of the generalized function algebra which are equal in distributional sense. This is the consequence of the continuity of $\xi(\cdot,\cdot)$ with respect to the second variable, in the sens of L^p-norm.

Concerning partial derivatives, for $u \in \mathcal{G}(\Omega, [L^2_\varepsilon(\mathfrak{O})])$ and $\alpha, k \in \mathbb{N}_0^d$, we have

$$\partial^\alpha m_{u_\varepsilon}(x) = m_{\partial^\alpha u_\varepsilon}(x), \quad x \in \Omega,\ \varepsilon \in I, \quad \text{and}$$
$$\partial_x^k \partial_y^k B_{u_\varepsilon}(x, y) = B_{\partial^k u_\varepsilon}(x, y), \quad x, y \in \Omega,\ \varepsilon \in I.$$

From

$$\sum_{i,j=1}^n B_{u_\varepsilon}(x_i - \tilde{x}_i, x_j - \tilde{x}_j) z_i \overline{z_j} = \sum_{i,j=1}^n E\big(u_\varepsilon(\omega, x_i - \tilde{x}_i)u_\varepsilon(\omega, x_j - \tilde{x}_j)\big) z_i \overline{z_j}$$
$$= E\left(\left|\sum_{j=1}^n u_\varepsilon(\omega, x_j - \tilde{x}_j) z_j\right|^2\right) \geqslant 0, \quad \varepsilon \in I,\ x_i, \tilde{x}_i \in \Omega,\ z_i \in \mathbb{C},\ i \in \mathbb{N},$$

it follows that the correlation function B_{u_ε} is positive definite for every $\varepsilon \in I$. Furthermore, the covariance function

$$C_{u_\varepsilon}(x, y) = B_{u_\varepsilon}(x, y) - m_{u_\varepsilon}(x)m_{u_\varepsilon}(y), \quad x, y \in \Omega,\ \varepsilon \in I,$$

is positive definite for every $\varepsilon \in I$. Indeed, since $m_{u_\varepsilon} = (E(u_\varepsilon(\omega, \cdot))$, we have

$$\sum_{i,j=1}^n C_{u_\varepsilon}(x_i - \tilde{x}_i, x_j - \tilde{x}_j) z_i \overline{z_j} = E\left(\left|\sum_{j=1}^n \big(u_\varepsilon(\omega, x_j - \tilde{x}_j) - m_{u_\varepsilon}(x_j - \tilde{x}_j)\big) z_j\right|^2\right) \geqslant 0.$$

Example 4.1. Let $X_t(\omega)$, $t \in \Omega$, $\omega \in \mathfrak{O}$, be an $L^2(\mathfrak{O})$ GFA-stochastic process with smooth sample functions. The corresponding element of $\mathcal{G}(\Omega, L^2(\mathfrak{O}))$, $Cd(X_t) = u$ has a representative $u_\varepsilon(\omega, t) = X_t(\omega)$, $t \in \Omega$, $\omega \in \mathfrak{O}$, $\varepsilon \in I$, as well as the following one:

$$\big(u_\varepsilon(t)\big)_\varepsilon = \big(\kappa_\varepsilon(t)X_t * \varphi_\varepsilon(t)\big)_\varepsilon = \left(\int_\Omega \big(\kappa_\varepsilon(s)X_s\big)\varphi_\varepsilon(s - t)\, ds,\right)_\varepsilon, \quad t \in \Omega, \tag{3}$$

where $(\varphi_\varepsilon)_\varepsilon$ denotes the net of mollifiers and $(\kappa_\varepsilon)_\varepsilon$ is the net introduced in the previous section.

If $X_t(\omega)$, $t \in \Omega$, $\omega \in \mathfrak{O}$, is an $L^2(\mathfrak{O})$ GFA-stochastic process with continuous sample functions, with expectation $m(t)$ and correlation function $B(t, s)$, then for $Cd(X_t) = u \in \mathcal{G}(\Omega, L^2(\mathfrak{O}))$ defined by its representative (3) we have

$$m_{u_\varepsilon}(x) = (\kappa_\varepsilon m)(x) * \varphi_\varepsilon(x), \quad x \in \Omega,\ \varepsilon \in I,$$

$$B_{u_\varepsilon}(x,y) = \kappa_\varepsilon(x)\kappa_\varepsilon(y)B(x,y) * \varphi_\varepsilon(x)\varphi_\varepsilon(y), \quad x,y \in \Omega,\ \varepsilon \in I,$$

and $[(m_{u_\varepsilon})_\varepsilon] \in \mathcal{G}(\Omega)$, $[(B_{u_\varepsilon})_\varepsilon] \in \mathcal{G}(\Omega \times \Omega)$. Indeed,

$$m_{u_\varepsilon}(x) = E\big(u_\varepsilon(\omega,x)\big) = \int_{\mathfrak{O}} u_\varepsilon(\omega,x)\,dP(\omega) = \int_{\mathfrak{O}} \bigg(\int_{\Omega} \kappa_\varepsilon(s)X_s(\omega)\varphi_\varepsilon(s-x)\,ds\bigg)\,dP(\omega)$$
$$= \int_{\Omega} \kappa_\varepsilon(s)\varphi_\varepsilon(s-x)\bigg(\int_{\mathfrak{O}} X_s(\omega)\,dP(\omega)\bigg)\,ds = \int_{\Omega} \kappa_\varepsilon(s)\varphi_\varepsilon(s-x)m(s)\,ds = \kappa_\varepsilon m)(x) * \varphi_\varepsilon(x).$$

For $B_{u_\varepsilon}(x,y)$ the proof is similar. Note that since $B(x,y)$ is positive definite, its Fourier transform $\hat{B}(\xi,\eta)$ is a positive distribution. Also, φ_ε is positive definite by construction. Thus, $B(x,y) * \varphi_\varepsilon(x)\varphi_\varepsilon(y)$ has the Fourier transformation $\hat{B}(\xi,\eta)\hat{\varphi}_\varepsilon(\xi)\hat{\varphi}_\varepsilon(\eta) \geqslant 0$ and therefore, $B_{u_\varepsilon}(x,y)$ is positive definite.

Remark. In [16] the notions of positive definiteness and weak positive definiteness of Colombeau functions are considered, and it is proved that a distribution $f \in \mathcal{D}'(\Omega)$ is positive definite if and only if the corresponding Colombeau generalized function $Cd(f)$ is weakly positive definite. However, we assumed that the net of mollifier functions φ_ε, $\varepsilon \in I$ is positive definite (which is not a strict restriction) in order to obtain positive definiteness in the strict sense of the correlation function of GFA-stochastic processes. This is important in order to have a correspondence between positive definite functions and Fourier transforms of positive tempered distributions (Bochner's theorem) which will implicitly be used in Theorem 4.3, while for weak positivity and weak positive definiteness it is still an open question (cf. [16]). Here, with the embedding $f \mapsto \kappa_\varepsilon f * \varphi_\varepsilon$ we have that f is a positive definite distribution (i.e. $\langle f, \theta * \theta^*\rangle \geqslant 0$, $\theta^*(x) = \overline{\theta(-x)}$, for all $\theta \in \mathcal{D}(\Omega)$) if and only if $f_\varepsilon = \kappa_\varepsilon f * \varphi_\varepsilon$ is a positive definite function (i.e. $\sum_{i,j=1}^n f_\varepsilon(x_i - y_i)z_i\overline{z_j} \geqslant 0$ for all $n \in \mathbb{N}$, $x_1, y_1, \ldots, x_n, y_n \in \Omega$, $z_1, \ldots, z_n \in \mathbb{C}$) for each $\varepsilon \in I$.

4.2. Gaussian $[L^2]$ GFA-stochastic processes

We keep in mind Gaussian generalized stochastic processes, and wish to define Gaussian GFA-stochastic processes.

Definition 4.1. Let $u \in \mathcal{G}(\Omega, [L^2_\varepsilon(\mathfrak{O})])$. It is said that u is a Gaussian GFA-stochastic processes, if there exists a representative $(u_\varepsilon)_\varepsilon$ and $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$, u_ε is a classical Gaussian stochastic process i.e. that for every $\varepsilon < \varepsilon_0$ and arbitrary $x_1, \ldots, x_n \in \mathbb{R}$, the probability that the random variable

$$X_\varepsilon = \big(u_\varepsilon(x_1,\omega), \ldots, u_\varepsilon(x_n,\omega)\big)$$

belongs to a Borel set $B \subset \mathbb{R}^n$ is

$$P(X_\varepsilon \in B) = \int_B \bigg(\frac{\det A_\varepsilon}{(2\pi)^n}\bigg)^{1/2} \exp\bigg(-\frac{1}{2}\langle A_\varepsilon t, t\rangle\bigg)\,dt, \quad \varepsilon < \varepsilon_0,$$

where A stands for a non-degenerate positive definite matrix, and

$$\langle A_\varepsilon t, t\rangle = \sum_{i=1}^n \sum_{j=1}^n a_{ij\varepsilon} t_i t_j, \quad \varepsilon < \varepsilon_0.$$

We will call $(u_\varepsilon)_\varepsilon$ a Gaussian representative of u. Also, instead of $\varepsilon < \varepsilon_0$ we will write $\varepsilon \in I$.

Theorem 4.1. *Let u be a Gaussian GFA-stochastic processes with Gaussian representative $(u_\varepsilon)_\varepsilon$ and $(B_{u_\varepsilon})_\varepsilon$ be a representative of its correlation function. Then, for all $x_1, \ldots, x_n \in \mathbb{R}$,*

$$A_\varepsilon = \big(B_{u_\varepsilon}(x_i, x_j)\big)^{-1}, \quad \varepsilon \in I.$$

Proof. Fix $x_i, x_j \in \Omega$. We have that

$$b_{i,j,\varepsilon} = B_{u_\varepsilon}(x_i, x_j) = E\big(u_\varepsilon(x_i,\omega)u_\varepsilon(x_j,\omega)\big), \quad x_i, x_j \in \Omega,\ \varepsilon \in I.$$

To prove the assertion, we will calculate $E(u_\varepsilon(x_i,\omega)u_\varepsilon(x_j,\omega))$, $x_i, x_j \in \Omega$, $\varepsilon \in I$, in another way. The random variable $u_\varepsilon(x_i)u_\varepsilon(x_j)$ (for fixed $\varepsilon \in I$) can be considered as a function of the n-dimensional random variable, whose distribution is given in Definition 4.1. Therefore ($\varepsilon \in I$),

$$E\big(u_\varepsilon(x_i,\omega)u_\varepsilon(x_j,\omega)\big) = \bigg(\frac{\det A_\varepsilon}{(2\pi)^n}\bigg)^{1/2} \int t_i t_j \exp\bigg(-\frac{1}{2}\langle A_\varepsilon t, t\rangle\bigg)\,dt = tr\big(E_{ij}A_\varepsilon^{-1}\big),$$

where E_{ij} denotes an $n \times n$-dimensional matrix whose elements all vanish except $e_{ij} = 1$, satisfying $\langle E_{ij}t, t\rangle = t_i t_j$. This gives $b_{i,j,\varepsilon} = tr(E_{ij}A_\varepsilon^{-1})$, $\varepsilon \in I$, and the proof is completed (cf. [3, p. 249], for explicit calculation). □

Theorem 4.2. *Partial derivatives of a Gaussian GFA-stochastic processes are again Gaussian GFA-stochastic processes.*

Proof. Let $u \in \mathcal{G}(\Omega, [L^2_\varepsilon(\mathfrak{O})])$ be a Gaussian GFA-stochastic processes with a Gaussian representative $(u_\varepsilon)_\varepsilon$, and let $\partial_{x_1} u$ be its $[L^2(\mathfrak{O})]$-derivative with a representative $(\partial_{x_1} u_\varepsilon(\omega, x))_\varepsilon$. The family $\mathcal{U}_\varepsilon = \{u_\varepsilon(\cdot, x)\colon x \in \Omega\}$, $\varepsilon \in I$, is a Gaussian family, and for $x \in \Omega$,

$$\partial_{x_1} u_\varepsilon(\cdot, x) = \lim_{h\to 0} \frac{u_\varepsilon(\cdot, (x_1 + h, \dots, x_n)) - u_\varepsilon(\cdot, x)}{h}, \quad \varepsilon \in I,$$

in the sense of $L^2_\varepsilon(\mathfrak{O})$-convergence. So we observe the linear closure in the sense of L^2-convergence of the Gaussian family $\mathcal{U}_\varepsilon$, denoted by $\overline{\mathcal{U}}_\varepsilon$. The family $\{\partial_{x_1} u_\varepsilon(\cdot, x)\colon x \in \Omega\}$ is a subfamily of $\overline{\mathcal{U}}_\varepsilon$, and therefore a Gaussian family as well. The proof for $(\partial^\alpha u_\varepsilon)_\varepsilon$ follows easily. □

The following theorem gives the complete characterization of Gaussian GFA-stochastic processes.

Theorem 4.3. *For given $m = [(m_\varepsilon(x))_\varepsilon] \in \mathcal{G}(\Omega)$ and $B = [(B_\varepsilon(x, y))_\varepsilon] \in \mathcal{G}(\Omega \times \Omega)$ such that the covariance function $C = [(C_\varepsilon(x, y))_\varepsilon] \in \mathcal{G}(\Omega \times \Omega)$ is positive definite (C_ε are positive definite), there exists a Gaussian GFA-stochastic processes $u \in \mathcal{G}(\Omega, [L^2_\varepsilon(\mathfrak{O})])$ with a Gaussian representative $(u_\varepsilon)_\varepsilon$, whose expectation and covariance function are m and C.*

Proof. Fix $\varepsilon \in I$. For any $n \in \mathbb{N}$ and arbitrary $x_1, \dots, x_n \in \Omega$, let $(c_{ij\varepsilon})$ be a matrix with elements

$$c_{ij\varepsilon} = C_\varepsilon(x_i, x_j) = B_\varepsilon(x_i, x_j) - m_\varepsilon(x_i)m_\varepsilon(x_j), \quad 1 \leqslant i, j \leqslant n.$$

As the covariance function C is positive definite and symmetric, the matrix $(c_{ij\varepsilon})$ is a positive definite and symmetric representative. Thus, for each $n \in \mathbb{N}$, we have an n-dimensional Gaussian distribution function, whose expectation and covariance matrix are $m_\varepsilon(x_1), \dots, m_\varepsilon(x_n)$ and $(c_{ij\varepsilon})$, respectively. The corresponding characteristic function is given by

$$(t_1, \dots, t_n) \rightsquigarrow \exp\left(-\frac{1}{2}\sum_{i,j=1}^{n}\big(B_\varepsilon(x_i, x_j) - m_\varepsilon(x_i)m_\varepsilon(x_j)\big)t_i t_j + i\sum_{i=1}^{n} m_\varepsilon(x_i)t_i\right).$$

For each fixed $\varepsilon \in I$, the family of n-dimensional Gaussian distribution functions (as n goes through $\mathbb{N}$) is compatible. This means that the conditions of the Kolmogorov theorem are satisfied, and there exists a probability space $(\mathfrak{O}, \mathfrak{U}_\varepsilon, P_\varepsilon)$ and a real Gaussian stochastic process $\{X_\varepsilon(x, \omega)\colon x \in \Omega,\ \omega \in \mathfrak{O}\}$ on it, whose expectation and covariance function are

$$x \rightsquigarrow m_\varepsilon(x), \qquad (x, y) \rightsquigarrow C_\varepsilon(x, y).$$

The same arguments yield that for every $k \in \mathbb{N}_0^d$, $|k| > 0$,

$$\partial^k m_\varepsilon(x) \quad \text{and} \quad \frac{\partial^{2k}}{\partial_x^k \partial_y^k} C_\varepsilon(x, y)$$

define a Gaussian stochastic process $\{Y_{\varepsilon,k}(x, \omega)\colon x \in \Omega,\ \omega \in \mathfrak{O}\}$ whose expectation and covariance function are exactly the given ones. Moreover, we have $\partial^k X_\varepsilon(x, \omega) = Y_{\varepsilon,k}(x, \omega)$ in the sense of $L^2_\varepsilon(\mathfrak{O})$-convergence, for every $\varepsilon \in I$. This shows that $X_\varepsilon(x, \omega)$ determines an $L^2_\varepsilon(\mathfrak{O})$ $\mathcal{C}^\infty$-stochastic process.

Since $B_\varepsilon = B_{X_\varepsilon}$, $\varepsilon \in I$, and $B = [(B_\varepsilon(x, y))_\varepsilon] \in \mathcal{G}(\Omega \times \Omega)$ it follows that for any compact set $K \times K \subset \Omega \times \Omega$, and any $k \in \mathbb{N}_0^d$, there exists an $a > 0$, such that

$$\sup_{(x,y)\in K\times K} \partial_x^k \partial_y^k B_\varepsilon(x, y) = \mathcal{O}\big(\varepsilon^{-a}\big) \quad \text{as } \varepsilon \to 0.$$

Using

$$\partial_x^k \partial_y^k B_{X_\varepsilon}(x, y)|_{(x,y)=(x,x)} = E\big(\partial_x^k X_\varepsilon(x, \omega)\partial_y^k X_\varepsilon(y, \omega)\big)\big|_{(x,y)=(x,x)} = \int_{\mathfrak{O}} \big|\partial_x^k X_\varepsilon(x, \omega)\big|^2\, dP_\varepsilon(\omega), \quad \varepsilon \in I,$$

we obtain that for every $K \Subset \Omega$ and $k \in \mathbb{N}_0^d$ there exists $a > 0$ such that

$$\sup_{x\in K} \int_{\mathfrak{O}} \big|\partial^k X_\varepsilon(x, \omega)\big|^2\, dP_\varepsilon(\omega) = \mathcal{O}\big(\varepsilon^{-a}\big) \quad \text{as } \varepsilon \to 0.$$

Thus we conclude that $[(X_\varepsilon(x, \omega))_\varepsilon]$ is a Gaussian GFA-stochastic process. □

This leads to:

Corollary 4.1. *Let $u \in \mathcal{G}(\Omega, [L^2_\varepsilon(\mathfrak{O})])$ be a GFA-stochastic process with expectation $m = [(m_{u_\varepsilon}(x))_\varepsilon] \in \mathcal{G}(\Omega)$ and a correlation function $B = [(B_{u_\varepsilon}(x, y))_\varepsilon] \in \mathcal{G}(\Omega \times \Omega)$. There exists a Gaussian GFA-stochastic process with the given expectation and correlation function.*

Example 4.2. Brownian motion $b \in \mathcal{G}(\mathbb{R}, [L^2_\varepsilon(\mathfrak{O})])$ is a Gaussian GFA-stochastic process with zero expectation and with correlation function

$$B_{b_\varepsilon}(x, y) = \min\{s, t\} * \varphi_\varepsilon(x)\varphi_\varepsilon(y) = \iint \min\{s, t\}\varphi_\varepsilon(s - x)\varphi_\varepsilon(t - y)\,ds\,dt, \quad x, y \in \mathbb{R}, \ \varepsilon \in I.$$

Example 4.3. White noise w is in [3] defined as the Gaussian generalized stochastic process with zero expectation and correlation function $B_w(\phi, \psi) = \int \phi(s)\psi(s)\,ds$, $\phi, \psi \in \mathcal{D}(\mathbb{R})$. According to the kernel theorem, for a positive definite functional B there exists a unique distribution $F \in \mathcal{D}'(\mathbb{R} \otimes \mathbb{R})$ such that $B(\phi, \psi) = \langle F, \phi\psi\rangle$. For white noise this distribution is $F_w(x, y) = \delta(x - y)$, $x, y \in \mathbb{R}$.

Now, in order to define white noise as a Gaussian GFA-stochastic process in $\mathcal{G}(\mathbb{R}, [L^2_\varepsilon(\mathfrak{O})])$, we can use different regularizations of the Dirac delta distribution. One possibility (following Example 4.1) is to put

$$B_{w_\varepsilon}(x, y) = \delta(s - t) * \varphi_\varepsilon(x)\varphi_\varepsilon(y) = \int \varphi_\varepsilon(s - x)\varphi_\varepsilon(s - y)\,ds, \quad x, y \in \mathbb{R}, \ \varepsilon \in I. \tag{4}$$

Since $B_{\partial b_\varepsilon}(x, y) = \partial_x\partial_y B_{b_\varepsilon}(x, y) = \partial_x\partial_y(\min\{s, t\} * \varphi_\varepsilon(x)\varphi_\varepsilon(y)) = \delta(s - t) * \varphi_\varepsilon(x)\varphi_\varepsilon(y) = B_{w_\varepsilon}(x, y)$ and each Gaussian GFA-stochastic process is uniquely determined with its expectation and correlation, we have that $w = \partial_x b$ i.e. white noise is the distributional derivative of Brownian motion.

Another possibility to regularize $\delta(x - y)$ is to put

$$\tilde{B}_{w_\varepsilon}(x, y) = \varphi_\varepsilon(x - y), \quad x, y \in \mathbb{R}, \ \varepsilon \in I, \tag{5}$$

and thus define white noise as the Gaussian GFA-stochastic process with zero expectation and correlation function $\tilde{B}$.

It is easy to show that B and $\tilde{B}$ are associated in $\mathcal{G}(\mathbb{R}^2)$, more precisely they both have the distributional shadow $\delta(x - y) \in \mathcal{D}'(\mathbb{R}^2)$. Thus, they determine Gaussian GFA-stochastic processes which are associated as elements of $\mathcal{G}(\mathbb{R}, [L^2_\varepsilon(\mathfrak{O})])$.

Example 4.4. The Gaussian GFA-stochastic process $u \in \mathcal{G}(\mathbb{R}, [L^2_\varepsilon(\mathfrak{O})])$ with zero expectation and with correlation function

$$B_{u_\varepsilon}(x, y) = \varphi^2_\varepsilon(x - y), \quad x, y \in \mathbb{R}, \ \varepsilon \in I,$$

is an example of a Gaussian GFA-stochastic process which is not associated with any element of $L(\mathcal{D}(\mathbb{R}), [L^2_\varepsilon(\mathfrak{O})])$, i.e. it does not have a distributional shadow.

4.3. Applications: Gaussian solutions of SPDEs

We give a simple general method of solving SPDEs in the framework of generalized Gaussian stochastic processes. This method can be applied also in the case of classical stochastic processes. Also, we formulate necessary conditions for the solvability of a SPDE in the space of Gaussian GFA-stochastic processes.

Let

$$P(x, D) = \sum_{|\alpha| \leqslant m} a_\alpha(x)\frac{\partial^\alpha}{\partial x^\alpha}, \quad a_\alpha \in \mathcal{G}(\Omega), \ \alpha \in \mathbb{N}^d_0, \tag{6}$$

be a linear differential operator with generalized coefficients. Recall [14], it is given by a representative $P_\varepsilon(x, D) = \sum_{|\alpha| \leqslant m} a_{\alpha,\varepsilon}(x)\frac{\partial^\alpha}{\partial x^\alpha}$, $a_{\alpha,\varepsilon} \in \mathcal{E}_M(\Omega)$, $\alpha \in \mathbb{N}^d_0$, $\varepsilon \in I$, and if $(a_{\alpha,\varepsilon})_\varepsilon - (\tilde{a}_{\alpha,\varepsilon})_\varepsilon \in \mathcal{N}(\Omega)$, $\alpha \in \mathbb{N}^d_0$, then $\tilde{P}_\varepsilon(x, D) = \sum_{|\alpha| \leqslant m} \tilde{a}_{\alpha,\varepsilon}(x)\frac{\partial^\alpha}{\partial x^\alpha}$, $\varepsilon \in I$, determines the same operator $P(x, D)$.

Consider the stochastic differential equation

$$P(x, D)u(\omega, x) = h(\omega, x), \quad x \in \Omega, \ \omega \in \mathfrak{O}, \tag{7}$$

where $h = [(h_\varepsilon)_\varepsilon]$ is a Gaussian GFA-stochastic process with expectation $m_h = [(m_{h_\varepsilon})_\varepsilon]$ and correlation $B_h = [(B_{h_\varepsilon})_\varepsilon]$. We interpret Eq. (7) as a family of equations

$$P_\varepsilon(x, D)u_\varepsilon(\omega, x) = h_\varepsilon(\omega, x), \quad x \in \Omega, \ \omega \in \mathfrak{O}, \ \varepsilon \in I, \tag{8}$$

in $\mathcal{E}_M(\Omega, [L^2_\varepsilon(\mathfrak{O})])$.

We want to find a solution $(u_\varepsilon)_\varepsilon$ to (8) so that it is a Gaussian representative of a Gaussian GFA-stochastic process u which solves (7).

Taking expectation on both sides of (8) we obtain

$$P_\varepsilon(x, D)m_{u_\varepsilon}(x) = m_{h_\varepsilon}(x), \quad x \in \Omega, \ \varepsilon \in I. \tag{9}$$

Now, we multiply (8) by $P_\varepsilon(y,D)u_\varepsilon(\omega,y)$ to have

$$P_\varepsilon(x,D)P_\varepsilon(y,D)u_\varepsilon(\omega,x)u_\varepsilon(\omega,y)=h_\varepsilon(\omega,x)h_\varepsilon(\omega,y),\quad x,y\in\Omega,\ \omega\in\mathfrak{O},\ \varepsilon\in I,$$

and then take expectation on both sides to obtain

$$P_\varepsilon(x,D)P_\varepsilon(y,D)B_{u_\varepsilon}(x,y)=B_{h_\varepsilon}(x,y),\quad x,y\in\Omega,\ \varepsilon\in I. \tag{10}$$

Assuming that Eqs. (9) and (10) have solutions m_{u_ε} and B_{u_ε}, $\varepsilon\in I$, in $\mathcal{E}_M(\Omega)$ and $\mathcal{E}_M(\Omega\times\Omega)$ respectively, and that B_{u_ε} is positive definite, there exists a Gaussian GFA-stochastic process u_ε determined by m_{u_ε} and B_{u_ε}.

Example 4.5. Let $\Omega=\mathbb{R}^n$. Consider now the equation

$$P(D)u(\omega,x)=h(\omega,x),\quad x\in\Omega,\ \omega\in\mathfrak{O},$$

where $P(D)$ is a differential operator with constant coefficients

$$P(D)=\sum_{|\alpha|\leqslant m}a_\alpha\frac{\partial^\alpha}{\partial x^\alpha} \tag{11}$$

and h is a Gaussian generalized functional stochastic process i.e. an element of $L(\mathcal{D}(\Omega),L^2(\mathfrak{O}))$ determined by the distributions $m_h\in\mathcal{D}'(\mathbb{R}^n)$ and $B_h\in\mathcal{D}'(\mathbb{R}^{2n})$. Applying the same method as above we obtain the system of equations

$$P(D)m_u(x)=m_h(x),\qquad P(D_x)P(D_y)B_u(x,y)=B_h(x,y),\quad x,y\in\mathbb{R}^n.$$

It is known (see [8]) that there exist distributions $m_u\in\mathcal{D}'(\mathbb{R}^n)$ and $B_u\in\mathcal{D}'(\mathbb{R}^{2n})$ that solve these equations. If B_u is positive definite in the sense of distributions, we obtain a generalized functional Gaussian stochastic process u with expectation m_u and correlation B_u that solves the initial equation. Using the embedding (2) we obtain the corresponding Gaussian GFA-stochastic process.

We continue to consider the classical differential operator with constant coefficients given in (11) and its formal Fourier transformation

$$\tilde{P}(t)=\sum_{|\alpha|\leqslant m}a_\alpha(-it)^\alpha,\quad t\in\mathbb{R}^n. \tag{12}$$

We will formulate a necessary condition for the existence of a Gaussian solution of (7). Applying the Fourier transformation to (10) we obtain

$$P(\xi)P(\eta)\hat{B}_{u_\varepsilon}(\xi,\eta)=\hat{B}_{h_\varepsilon}(\xi,\eta),\quad \xi,\eta\in\Omega,\ \varepsilon\in I. \tag{13}$$

Since B_{h_ε} is a correlation function and thus positive definite, $\hat{B}_{h_\varepsilon}$ is a positive distribution for all $\varepsilon\in I$. (This means that for every $\varepsilon>0$, $\langle\hat{B}_{u_\varepsilon},\phi\rangle\geqslant 0$ for all $\phi\in\mathcal{D}(\mathbb{R}^{2n})$ with the property $\phi\geqslant 0$.) If there exists a solution B_{u_ε}, it should also be positive definite in order to be a well-defined correlation function. Note, in [3] it is proved that if $B_u(x,y)$ is positive definite, then $P(D_x)P(D_y)B_u(x,y)$ is also positive definite. But the converse is generally not true, and this is what we are interested in: what conditions are necessary to hold for the operator $P(D)$ in relation to B_{h_ε} in order that the solution B_{u_ε} is positive definite? Assuming that there exists a nonnegative solution $\hat{B}_{u_\varepsilon}$ of (13) it follows: If $\hat{B}_{h_\varepsilon}=0$, then $\tilde{P}(\xi)\tilde{P}(\eta)=0$ or $\hat{B}_{u_\varepsilon}=0$. On the other hand if $\hat{B}_{h_\varepsilon}>0$ ($\hat{B}_{h_\varepsilon}\geqslant 0$ and $\hat{B}_{h_\varepsilon}\neq 0$), then $\hat{B}_{u_\varepsilon}>0$ and $\tilde{P}(\xi)\tilde{P}(\eta)\geqslant 0$. Thus, it is necessary that

$$\bigl\{(\xi,\eta)\in\mathbb{R}^{2n}:\ \hat{B}_{h_\varepsilon}(\xi,\eta)=0\bigr\}\subseteq\bigl\{(\xi,\eta)\in\mathbb{R}^{2n}:\ \tilde{P}(\xi)\tilde{P}(\eta)=0\bigr\}\cup\bigl\{(\xi,\eta)\in\mathbb{R}^{2n}:\ \hat{B}_{u_\varepsilon}(\xi,\eta)=0\bigr\}$$

and

$$\bigl\{(\xi,\eta)\in\mathbb{R}^{2n}:\ \hat{B}_{h_\varepsilon}(\xi,\eta)>0\bigr\}\subseteq\bigl\{(\xi,\eta)\in\mathbb{R}^{2n}:\ \tilde{P}(\xi)\tilde{P}(\eta)\geqslant 0\bigr\}\cap\bigl\{(\xi,\eta)\in\mathbb{R}^{2n}:\ \hat{B}_{u_\varepsilon}(\xi,\eta)>0\bigr\}.$$

Note that we assume the positivity in a distributional sense; it is known, if a distribution is a continuous function, then this notion reduces to the usual notion of a positive function.

We summarize the results of these considerations in the following proposition.

Proposition 3. *Let $\Omega=\mathbb{R}^n$. The SPDE*

$$P(D)u(\omega,x)=h(\omega,x),\quad x\in\Omega,\ \omega\in\mathfrak{O},$$

where $h=[(h_\varepsilon)_\varepsilon]$ is a Gaussian GFA-stochastic process with correlation $B_h=[(B_{h_\varepsilon})_\varepsilon]$, and $P(D)$ is a differential operator with constant coefficients as in (11)*, has a Gaussian solution $u=[(u_\varepsilon)_\varepsilon]\in\mathcal{G}(\Omega,L^2(\mathfrak{O}))$, if the solutions B_{u_ε}, $\varepsilon\in I$, of the differential equations*

$$P(D_x)P(D_y)B_{u_\varepsilon}(x,y) = B_{h_\varepsilon}(x,y), \quad x,y \in \Omega,\ \varepsilon \in I,$$

determine an element $[(B_{u_\varepsilon})_\varepsilon] \in \mathcal{G}(\Omega \times \Omega)$ *and moreover for every* $\varepsilon \in I$, B_{u_ε} *are positive definite.*

A necessary condition for the positive definiteness of B_{u_ε}, $\varepsilon \in I$, *is that the differential operator* $P(D)$ *and* B_{h_ε}, $\varepsilon \in I$, *satisfy*

$$\{(\xi,\eta) \in \mathbb{R}^{2n}\colon\ \hat{B}_{h_\varepsilon}(\xi,\eta) > 0\} \subseteq \{(\xi,\eta) \in \mathbb{R}^{2n}\colon\ \tilde{P}(\xi)\tilde{P}(\eta) \geqslant 0\},$$

where $\tilde{P}(t)$, $t \in \mathbb{R}^n$, *is the formal Fourier transformation of* $P(D)$ *defined as in* (12)*.*

Example 4.6. Consider the equation $u'(\omega,x) = w(\omega,x)$, where w is the white noise Gaussian GFA-stochastic process defined in Example 4.3. Then, (9) reduces to $m'_{u_\varepsilon}(x) = 0$, while (10) reduces to $\partial_x\partial_y B_{u_\varepsilon}(x,y) = \int \varphi_\varepsilon(s-x)\varphi_\varepsilon(s-y)\,ds$. It was shown in Example 4.3 that the latter equation has a solution $B_{u_\varepsilon}(x,y) = \min\{x,y\} * \varphi_\varepsilon(x)\varphi_\varepsilon(y)$, while the first equation has a solution e.g. $m_{u_\varepsilon}(x) = 0$. (Note, that uniqueness cannot be obtained in this way.) Thus, a solution of $u'(\omega,x) = w(\omega,x)$ is $u = b$, where b is the Brownian motion process from Example 4.3. Applying the Fourier transformation to $\partial_x\partial_y B_{u_\varepsilon}(x,y) = \delta(x-y) * \varphi_\varepsilon(x)\varphi_\varepsilon(y)$ we obtain

$$-\xi\eta\hat{B}_{u_\varepsilon}(\xi,\eta) = \delta(\xi+\eta)\hat{\varphi}_\varepsilon(\xi)\hat{\varphi}_\varepsilon(\eta).$$

Indeed, we have that the necessary condition:

$$\{(\xi,\eta) \in \mathbb{R}^2\colon\ \xi = -\eta\} \subseteq \{(\xi,\eta) \in \mathbb{R}^2\colon\ \xi\eta \leqslant 0\}$$

is satisfied.

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